Two Essays on Intergenerational Linkage*

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“Some Propositions on Intergenerational Risk Sharing, Social Security and Self-Insurance”

Abstract

This article describes, within a microeconomic intergenerational bargaining framework incorporating two discrete periods and binary states of risks, some new aspects regarding the mixture of intergenerational risk sharing and social security. Here, state-dependent utility under mortality risk proves to generate parents’ peculiar indifference curve regarding insurance contract, and self-insurance is shown to play a crucial role on the decision regarding social security holding and intergenerational transfer contract. This peculiar aspect, given for the first time in this article, also derives some novel features of insurance theory under lifetime uncertainty, where the current position in social security contract could adversely affect parents’ decision regarding intergenerational risk sharing with children. In addition, other basic results regarding the sensitivity to default risk and taxation, and furthermore some implications in unfunded social security are summarized.

1. Introduction

The objective of this article is simply and clearly to describe some new economic aspects of intergenerational risk sharing under lifetime uncertainty within a microeconomic bargaining framework. Atkinson and Stiglitz (1980) and Obstfeld and Rogoff (1996) are textbooks of public economics and international macroeconomics, especially the latter of which contains the description of a risk sharing with default risk and saving. For some other examples among related literatures, Shiller (1999), Ball and Mankiw (2001), Enders and Lapan (1982) and Gordon and Varian (1988) examine the economic role of intergenerational risk sharing. Yaari (1965) is a classical article, which pursuits the implications of life insurance under the mortality risk. Hayashi, Altonji and Kotlikoff (1996) is an empirical work on intra-gamily & inter-generational risk sharing accompanying the possibility of self-insurance. Analyses on bequest motives appear, among many, in Abel (1985, 1987), Hurd (1989), Bernheim, Shleifer and Summers (1985) and Bernheim (1991). Especially Abel

(1987) does it from two-sided altruistic framework, and Bernheim et al. (1985) do by taking a bequest as a "threat of retaliation" strategy or a "reward of children's egoistic action" by parents. Kotlikoff and Spivak (1981) discuss the role of family as incomplete annuity insurance, and Kotlikoff, Shoven and Spivak (1986) examine the effects of market annuity insurance on saving and inequality. Altig and Davis (1993) analyze, in two-sided altruistic economy, the effect of funded and unfunded social security from the viewpoint of borrowing constraints.

One important missing point in the above literature is to analyze the effect of the bargaining on the whole economy, from comprehensive viewpoints in the possible mixture of three forms of insurance, intra-family transfer, unfunded social security and finally self insurance. Another, still controversial, is to specify the attribute of intergenerational bargaining between young adults and old retired people under life uncertainty (for example, is it motivated on altruism? or competitiveness? or other mechanisms?) In spite of some pioneering papers, beginning from Yaari's work (1965) and going through some by Kotlikoff et al. (1981, 1986), there seems to me still to exist some room for further elaboration. The main reason is that none of the literature seems to be explicitly conscious of the shapes of indifference curve and corresponding Pareto optimal contract curve generated on the existence of morality risk, and without the clear awareness of them, any concrete implications regarding incentives to hold any (all or some) forms of insurance would not be able to be successfully extracted. Furthermore, without these investigations implemented as rigorously as possible, any dynamic and macroeconomic analysis, whether based on altruism or competitiveness, would not be appropriately accomplished. This article, as the first step for attaining these objectives, tries to focus on some static environments between aged retiree exposed to mortality risk and young working adult to income risk, in which other life strategies, like saving, education or fertility, are assumed to be exogenously given. Instead of continuous time case, which unnecessarily complicates the analysis, I divide the corresponding stage discretely into two periods, where only the binary states of mortality/income risk exist and are revealed at the boundary of these two adjacent periods. Then I treat intergenerational support/bequest contracts as state variables, and draw Pareto optimal contract curves. Strikingly this simplified setting proves to derive out some crucial implications regarding how the two adjacent generations efficiently combine the three options of risk sharing.

This article is organized as follows. In section 2, a basic framework is set, in which two adjacent generations, parents in old adulthood and children in young adulthood are facing the decision regarding intergenerational risk sharing with/without an available old-age social security for parents. In section 3, some characteristics in parents' difference curve are explained, where self-insurance plays an important role on the insurance contract decision. On the basis of these analyses, I claim some fundamental propositions regarding the optimal allocation of social security and intergenerational risk sharing in section 4, and some regarding the sensitivity to default risk and taxation in social security in section 5. Finally in section 6, I apply the discussion to unfunded social security and summarize some implications.

2. Basic setting

Definitions of main notations and equations used in this article are summarized in

Table 1 and 2. At first I divide each generation's lifetime roughly into three stages, "Y" (for the infant), "M" (for the young adulthood) and "O" (for the old adulthood), each of which corresponds with each discrete period (≒30 years). As in the typical overlapping generation model, assume that, at the beginning of period t, two adjacent generations, "p" (parents) and "c" (children), are now going to begin stage O and M, respectively. Parents hold an available asset A^p, and children's disposable income is W^c. During this period, there exist two types of binary risk, the risk of death (mortality risk $s_1(=1,0)$) for parents, and that of disposable income $s_2(=1,0)$ for children. The risk of death exists is revealed exactly at the middle point of stage O, when they are alive $(s_1 = 1)$ with probability φ, or die $(s_1 = 0)$ with probability $1-\varphi$. The income risk is revealed exactly at the middle point of stage M, when they earn higher income W_1^c $(s_2 = 1)$ with probability φ', or lower income W_0^c $(s_2 = 0)$ with $1-\varphi'$. Therefore the revelation of mortality risk for parents, and that of income risk for children exactly coincide with each other in time. Each generation $i(=p,c)$ holds an egoistic utility, which depends explicitly only on its own consumption only during stage M and O, not during Y, and takes a form of *state-dependent utility*:

$$\tilde{u}^i = u(c^i_{M1}) + \beta u(c^i_{M2}) + \beta^2\{u(c^i_{O1}) + \beta u(c^i_{O2})\} \qquad \text{if } s_1 = 1$$

$$\tilde{u}^i = u(c^i_{M1}) + \beta u(c^i_{M2}) + \beta^2 u(c^i_{O1}) \qquad \text{if } s_1 = 0$$

$$u(c) = c^{1-\sigma}/(1-\sigma) \text{ if } \sigma \neq 1, \text{ or } u(c) = \ln c \text{ if } \sigma = 1 \qquad (2.1)$$

Here β is constant time preference for each half period, σ is constant relative risk aversion coefficient, c_{M1}^{i} is the real consumption of generation i during the first half period of stage M, c_{O2}^{i} is the real consumption of generation i during the second half period of stage O, or etc. Utility function $u(\cdot)$ is increasing and concave, and assumes ordinary Inada conditions. Real interest rate for each half period is denoted by r. Children's life strategies during stage M (number of children to bear N^c, human capital investment for each child H^c, asset plan A^c) are exogenously given, except for the intergenerational transfer contracts with parents (S,B). For parents, their asset holding A^p is already fixed and there are two options of old adulthood insurance for mortality risk, intergenerational transfer with children (S,B) and social security (R,P).[1] [2] Here the left-hand side of (S,B) and (R,P) denotes *receipt* conditional on $s_1=1$, and the right-hand side does *payment* on $s_1=0$, both sides measures by present value at the beginning of the period. If these transfer contracts are *actuarially fair*, they necessarily satisfy:

$$l_1 : \varphi S = (1-\varphi)B \quad \text{or} \quad l_1 : \varphi R = (1-\varphi)P \tag{2.2}$$

At first assume that s_1 and s_0 are uncorrelated. Then *the associated indirect utilities* for parents and children regarding a transfer contract schedule (S,B) are represented as:

$$\tilde{v}^{p}(S,B,\varphi,\beta,r,A^{p}) \equiv u(A^{p}-B)+\varphi\beta u((1+r)(S+B)) \tag{2.3a}$$

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[1] (S,B) and (R,P) denote $(Support, Bequest)$ and $(\mathrm{Re}ceipt, Payment)$, respectively.

[2] In this article I do not set any substantial distinction between social security and market insurance.

$$\tilde{v}^c(S,B,\varphi,\beta,r,W^c) \equiv \max_{c_1}\begin{pmatrix}\varphi[u(c_{M1})+\beta u((1+r)(W^c-S-c_{M1}))] \\ +(1-\varphi)[u(c_{M1})+\beta u((1+r)(W^c+B-c_{M1}))]\end{pmatrix} \quad (2.3b)$$

If social security program (R,P) is also available for parents, then (2.3a) becomes:

$$\tilde{v}^p(S+R,B+P,\varphi,\beta,r,A^p) \equiv u(A^p-B-P)+\varphi\beta u((1+r)(S+R+B+P)) \quad (2.4)$$

$W^c \equiv \varphi' W_1^c + (1-\varphi')W_0^c$ is defined as the expected disposable income, under no correlation between s_1 and s_0. Derivations of (2.3a, b) are as follows. From (2.1), parents' utility as of the beginning of stage O is $u(c_{O1}^p)+\beta u(c_{O2}^p)$ for $s_1 = 1$ and $u(c_{O1}^p)$ for $s_1 = 0$. Under transfer contract with children (S,B), their old age consumptions are confined to $c_{O1}^p = A^p - B$ and $c_{O2}^p = (1+r)(S+B)$. Then the expected utility with regard to mortality risk s_1 is $\varphi\{u(c_{O1}^p)+\beta u(c_{O2}^p)\}+(1-\varphi)u(c_{O1}^p)$, being equal to $u(A^p-B)+\varphi\beta u((1+r)(S+B))$ (2.3a). On the other hand, children's utility as of the beginning of stage M is $u(c_{M1}^c)+\beta u(c_{M2}^c)+\beta^2\tilde{v}^p(\cdots,A^c)$ (2.5), where $\tilde{v}^p(\cdots,A^c)$ is their old age indirect utility based on the assumptions that the whole old age life strategies including A^c is already fixed. Under transfer contract with parents (S,B), their middle age consumptions are confined to $c_{O2}^c = W^c - S - c_{M1}$ for $s_1 = 1$, and $c_{O2}^c = W^c + B - c_{M1}$ for $s_1 = 0$. Then taking the expectation of (2.5) as well as eliminating constant term $\beta^2\tilde{v}^p(\cdots,A^c)$ derives (2.3b).

The argument in this article, to repeat, assumes that parents and children limit their focus only on a myopic bargaining, standing at the beginning of stage O and M, respectively. Rigorously speaking children are supposed to set up their future life strategies (N^c, H^c, A^c), permissibly forthcoming state contingent, within their recursive dynamic programming over infinite time horizon. However, again reversely, without clarification of precise attributes in this short-run bargaining, any long run effects and implications of mixture in various assumed risk sharing would not be able to be precisely derived. And I am going to clarify them actually.

3. Some peculiar aspects of intergenerational risk sharing

See Figure 1. The dotted line a , c , and d are the indifference curves of parents, which draws the contour lines $\tilde{v}^p(S, B, \varphi, \beta, r, A^p) = \bar{v}^p$ for distinct constants, $\bar{v}^p$'s. On the other hand, b , e , and f are those of children, which draws the contour lines $\tilde{v}^c(S, B, \varphi, \beta, r, \overline{\overline{W}}^c) = \bar{v}^c$ for distinct $\bar{v}^c$'s. Under the settings of section 2, there exists a set of intergenerational transfer contract (S, B) , such that: (1) both parents and children are willing to conclude the contract. (Participation constraints), and (2) each contract is Pareto-optimum. (Pareto optimality conditions) Furthermore, the compact set denoted by $\underline{X}_p$, satisfying the above conditions (1) and (2), is located inside the area $S > 0$, $B > 0$ and $(1-\varphi)B > \varphi S$. This Pareto set $\underline{X}_p$ is a bold segment line GF , where point F is a tangent point of parents' indifference curve a , and that of children e , and point G is a tangent point of children's indifference curve b and that of parents d . Any indifference curves of parents (a , c or d) are shown to be tangent with two lines, $B = A^p$ and *certainty line* $l_2 : B = -S$. Children, if they do not conclude any transfer contract, will be undoubtedly at point O (that is, $(S, B) = (0,0)$), being tangent with the *actuarially fair line* $l_1 : \varphi S = (1-\varphi)B$. In general, as illustrated in the indifference curve f, any arbitrary indifference curve of children is tangent with the *constant premium line* $l_1' : (1-\varphi)B - \varphi S = k$, at the intersection of l_1' and $l_2 : B = -S$.

See Figure 2. Now I examine parents' position within a given transfer contract scheme (S, B) . If parents are not given any transfer contract, their position is illustrated as point $D : (S, B) = \left(0, A^p / (1 + \{\varphi\beta(1+r)^{1-\sigma}\}^{-1/\sigma})\right)$, where a is tangent with the horizontal axis $S = 0$, exactly at point D .[3] Here B is not an amount of bequest, but is some conditional cost on death ($s_1 = 0$) to be additionally discarded as a result of partially self-insuring mortality risk s_1 . Thus point D is an optimal "self-insurance (self-contract)", which parents would choose when the social security is not available. Instead, if parents make the actuarially fair and flexible social security contract, their position is point $E : (R, P) = \left((1-\varphi)\kappa A^p, \varphi\kappa A^p\right)$, where $\kappa = \{\beta(1+r)^{1-\sigma}\}^{1/\sigma} / (1 + \{\varphi^\sigma \beta(1+r)^{1-\sigma}\}^{1/\sigma})$ and their indifference curve c is tangent with $l_1 : \varphi R = (1-\varphi)P$, at point E .[4] See point

[3] In this case, parents' maximization problem is equivalent with maximizing their indirect utility associated with transfer contract (S, B) , (2.3a), with regard to B , keeping S fixed at 0.

[4] Note that parents' maximization problem is equivalent with maximizing their associated indirect utility (2.3a), with regard to (S, B) satisfying l_1 . Here intergenerational transfer (S, B) is

I and J, both located on l_1. I is the point where a, which passes point D, intersects with l_1. Therefore, I is a *reservation* actuarially fair contract, which assures a minimum utility, same as an "optimal self-insuring contract" D. On the other hand, J is a point on l_1, at which an indifference curve of parents takes a minimum in S exactly at J. Now I consider some fixed actuarially fair contract on l_1, represented as point $K:(R_K, P_K)$. Assume that K is located on between point O and J. In this case, parents can be even better off than at K, by discarding some additional cost P' say, conditionally on death (along the axis in B), as a kind of self-insuring contract. Let point K' be the tangent point of parents' indifference curve and $S = R_K$. Then the optimal additional cost P', which parents should discard conditionally on death, is calculated as distance KK'. If K is located in the upper-right of J along l_1, then parents do not have to pay any additional cost along in B. The overall locus of a mixed contract schedule $\widetilde{K}$ say, which should include that additional and conditional cost in correspondence with each given contract K, would be a semi-segment of line, $DJEQ$, as drawn in a bold line in Figure 2. I denote this set, which can be optimally attained as a result of making use only of an actuarially fair contract set, by $\underline{X}_{ac}$. Clearly $\widetilde{K}$ coincides with K' if K is located on a segment line OJ, and coincides with K itself if K is on a semi-segment line JQ

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replaced with the notation for social security contract (R, P).

Without any contracts concluded, parents would stand at point D, while children would at a different point O. This aspect makes it for both parents and children impossible to set initially some value for the state contingent claim between two states of s_1, or equivalently to set the initial relative price between S and B. This is a totally different point from Arrow-Debreu state-contingent exchange economy, in which state contingent claim (or state price) enables them to arrive at a market-clearing and Pareto optimal equilibrium.[5] Therefore, in this microeconomic bargaining frame work, an automatic price adjustment process to a unique equilibrium point on $\underline{X}_p$ cannot be expected, as far as any additional restriction (e.g., regarding the altruistic weight in utility between parents and children) or any other peculiar agreements or algorithms are not introduced.[6] This is one important economic feature of intergenerational contract curve $\underline{X}_p$. On the other hand, with a fixed level of available social security, for example $K = J$, in which self-insurance is not necessary, Arrow-Debreu state-contingent exchange economy can be well defined. In this case, an equilibrium (Pareto optimal and market clearing) contract does not depend on the existence of altruism between parents and children, since, in general, the weight of altruism does not transform the shape of *extended* contract curve, which is drawn just by relaxing

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5 If both parents and children agree with standing initially at point O, there does exist a competitive equilibrium on $\underline{X}_p$ (GF).

6 One example is *altruistic* utility of the form $U^p = \tilde{u}^p + \psi(\tilde{u}^c)$.

participation constraints.[7]

4. Some propositions regarding the mixture of intergenerational risk sharing and social security

Now I compare, within the current framework, actuarially fair social security and intergenerational transfer contract, from parents' viewpoint. Especially one important question is: Do parents choose only an actuarially fair social security or only an intergenerational transfer contract with children, or both of them? Although it depends on where an available social security K and an available intergenerational transfer Y are located on l_1 and on $\underline{X}_p$ respectively, some aspects regarding this question can be extracted, by setting one simple assumption regarding children's behavior, that they would accept any intergenerational transfer which is offered from parents, if it assures at least the same utility as at point Y in terms of children's associated indirect utility (2.3b). Denote parents' maximized utilities, which can be attained by concluding only social security K, only intergenerational transfer Y, and both of them, by $V^p(K)$, $V^p(Y)$ and $V^p(K+Y)$, respectively.[8] At first, I claim a following proposition and corollary. Proofs in this section are

[7] With a fixed social security (R_K, P_K), this extended contract curve (not $\underline{X}_p$) is drawn by a set of points, where parents' indifference curve is tangent with the curve generated by shifting children's indifference curve in parallel along with l_1 by vector (R_K, P_K).

[8] With each of these three options, parents may pay, if necessary, an additional and conditional cost

given in Appendix 1.

Proposition 1: See Figure 2. Then:

(i) Assume that Y coincides with G, the point which attains parents' maximum utility on $\underline{X}_p$. Then, for any arbitrary K, which is located on the segment line of l_1, OZ, it holds that $V^p(Y) \leq V^p(K+Y)$ and $V^p(K) \leq V^p(K+Y)$.

(ii) Assume that Y coincides with F, the point which attains parents' minimum utility on $\underline{X}_p$. Then, for any arbitrary K on l_1 such that $OK \leq O'Z'$, it holds that $V^p(Y) \leq V^p(K+Y)$ and $V^p(K) \geq V^p(K+Y)$.

Corollary 1: Consider an already concluded (mandatory) intergenerational transfer Y on $\underline{X}_p$. Then, any arbitrary social security K on l_1 ($S \geq 0, B \geq 0$) surely enhances parents' indirect utility without any necessity to discard any additional and conditional cost, if K is not extremely large in amount. This always holds whether there exists some correlation between parents' state s_1 and children's state s_2 or not.

Corollary 1 is clear from children's indifference curve under positive/negative correlation between s_1 and s_2, as shown in Figure 3 and 4. Proposition 1 has quite interesting economic implications. First, parents, together with a mandatory intergenerational transfer,

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along the axis in B (self-insurance), as explained in Section 3. For rigorous formulation of parents' problems to be solved, see Appendix 1.

would almost always choose to take any arbitrarily given social security. Second, but if reversely any social security is mandatory, while a fixed intergenerational transfer is not, it may not be the case. If a non-mandatory intergenerational transfer Y coincides with G, the maximum utility point, parents are very likely to take both of any arbitrary K and the intergenerational transfer $Y(=G)$, on the other hand, if a non-mandatory intergenerational transfer Y coincides with F, the minimum utility point, parents are very likely to take only social security for any arbitrary K. This implies that it is quite natural to think that for any arbitrary, but mandatory K, which is not extremely large in amount, there exists some point Y on $\underline{X}_p$, such that parents are indifferent to whether to accept an intergenerational contract or not. From continuity and monotonicity of parents' indirect utilities on $\underline{X}_p$, I have a proposition and a corollary as follows.

Proposition 2: See Figure 2. For any arbitrary social security K, which is located on l_1, such that $OK \le O'Z'$, there always exists at least one intergenerational transfer, $\overline{Y}(K)$ as a function of K on $\underline{X}_p$, such that $V^p(K) = V^p(K + \overline{Y}(K))$.

Corollary 2: Assume $\overline{Y}(K)$ is not point G. Then $\overline{Y}(K)$ moves slightly along $\underline{X}_p(GF)$ in the direction to G, for a slight positive change in K.

Just for purely mathematical interest, I claim following two lemmas.

Lemma 1: $\underline{X}_p$ (A segment line GF) has a negative tangent slope (of R with regard to P), which is less than -1. Also, a segment line JD, which is a part of $\underline{X}_{ac}$, has a negative tangent slope (of R with regard to P), which is less than -1.

Lemma 2: Denote a tangent point of parents' indifference curve with a constant premium line, $l_1' = (1-\varphi)P - \varphi R = k$, by point E^k. (So, $E^{k=0}$ is the same point as E.) Then the locus of the set of point E^k, has a negative tangent slope (of R with regard to P), which is less than -1.

The proof of next Proposition is directly derived from Lemma 2.

Proposition 3: Assume an already concluded (mandatory) intergenerational transfer Y on $\underline{X}_p$, and a flexible, actuarially fair social security K on l_1. Then the optimal social security $\hat{K}(Y)$ as a function of Y, which gives the maximum of parents' indirect utility $V^p(\hat{K}(Y)+Y)$, decreases in its size $O\hat{K}$, as Y moves along $\underline{X}_p$ from G to F.

Lastly I examine the simplest case in which only actuarially fair social security K on l_1 is available for parents. Assume that only actuarially fair social security K on l_1 is available for parents. Then, as K moves along l_1 from O to Q, that is, as $\tilde{K}$ moves along $\underline{X}_{ac}$ ($DJEQ$), parents' marginal utility of social security decreases. Especially at point J the marginal utility discontinuously jumps into a lower level, and it becomes 0 (zero) at point E. This aspect shows that if social security is some point between O and J, the marginal utility (benefit) of social security is relatively high because of the decreasing cost of self-insurance. Together with intergenerational transfer, however, this kind of discontinuity does not appear.

5. Other results regarding the sensitivity to default risk and taxation in social security

In this section I limit my analysis only on social security, and examine the sensitivity both of parents and children to default risk and taxation on the demand for social security, wherein now children's income risk s_2 arises and $W_1^c > W_0^c$.[9]

Sensitivity to default risk

I introduce another risk s_3 for the default risk of social security system, where $s_3 = 1,0$ represents non-default or default, revealed, as in s_1 and s_2, at the middle point of the underlying period. Also assume that the probability of default ($s_3 = 0$) is η, and s_3 has no correlation with s_1 and s_2, respectively.[10] At first, consider the demand for social

[9] Therefore no correlation condition of (2.3b) has been relaxed.

[10] It seems appropriate to assume that there exists no correlation among s_1, s_2 and s_3, so far as there does not occur any strong social systemic risk. Otherwise, these three risks may have a considerable strong *positive* correlation with each other.

security by parents during stage O, $(R^p, P^p)(=(R,P))$. The pay off of parents for each realization of two relevant risks, s_1 and s_3, is as following. Parents receive R for $\{s_1 = 1, s_3 = 1\}$ with probability $\varphi(1-\eta)$, $-P$ for $\{s_1 = 0, s_3 = 1\}$ with $(1-\varphi)(1-\eta)$, $0(zero)$ for $\{s_1 = 1, s_3 = 0\}$ with $\varphi\eta$, $-P$ for $\{s_1 = 0, s_3 = 0\}$ with $(1-\varphi)\eta$, respectively. In case of "default", parents still have a liability (P), if they die (that is, if $\{s_1 = 0, s_3 = 0\}$ occurs). Now I have two definitions for actuarially fair condition: Conditional actuarially fair condition on non-default, $l_1 : \varphi R = (1-\varphi)P$ (5.1), and unconditional actuarially fair condition, $l_1^\eta : \varphi(1-\eta)R = (1-\varphi)P$ (5.2). Furthermore, parents' associated indirect utility including default risk is re-defined as:

$$\tilde{\tilde{v}}^p(R,P,\varphi,\eta,\beta,r,A^p) \equiv u(A^p - P) + \varphi(1-\eta)\beta u((1+r)(R+P)) + \varphi\eta\beta u((1+r)P) \quad (5.3)$$

Now I examine the sensitivity of parents' demand for an actuarially fair social security in the sense of (5.1) and (5.2), when η deviates slightly from 0 (zero) by a positive bit. In particular, my interest is in the sensitivity of an optimal contract E and a reservation contract I, to η. In order to do this, I denote the tangent point of either l_1 or l_1^η with the indifference curve based on this "modified" associated indirect utility (5.3), by $E^\eta : (R_E(\eta), P_E(\eta))$. Also I denote the point on either l_1 or l_1^η, which, with the indifference curve based on a modified associated indirect utility, (5.3), attains the same utility as at point D of $\eta = 0$ ($D^{\eta=0}$, say), by $I^\eta : (R_I(\eta), P_I(\eta))$. Now I claim a following proposition. Proofs in this section are given in Appendix 2.

Proposition 4: (Parents' demand sensitivity to default risk) Assume that social security has default risk with conditional actuarially fair condition (5.1). Then I have: (i) $R_E'(\eta) > 0, P_E'(\eta) > 0$ if $\sigma > 1$, $R_E'(\eta) = 0, P_E'(\eta) = 0$ if $\sigma = 1$, and $R_E'(\eta) < 0, P_E'(\eta) < 0$ if $\sigma < 1$. Furthermore I have: (ii) $R_I'(0) > 0, P_I'(0) > 0$, irrespective of the value of σ.

Instead of (5.1), assume that social security has default risk with unconditional actuarially fair condition (5.2). Then it always holds that: (iii) $R_E'(\eta) > 0, P_E'(\eta) > 0$, and (iv) $R_I'(0) > 0, P_I'(0) > 0$, irrespective of the value of σ.

Next consider the optimal demand for social security by children during stage M, $(R^c, P^c)(=(R,P))$. The pay off of children for each realization of two relevant risks, s_2 and s_3, is as following. Children receive $-P$ for $(s_2 = 1, s_3 = 1)$ with probability $\varphi'(1-\eta)$, R for $\{s_2 = 0, s_3 = 1\}$ with $(1-\varphi')(1-\eta)$, $-P$ for $\{s_2 = 1, s_3 = 0\}$ with $\varphi'\eta$, $0(zero)$ for $\{s_2 = 0, s_3 = 0\}$ with $(1-\varphi')\eta$, respectively. In case of "default", children still have a liability (P), if they have a higher income (that is, if $\{s_2 = 1, s_3 = 0\}$ occurs). Now I have two definitions for actuarially fair condition: Conditional actuarially fair condition on non-default, $l_3 : \varphi' P = (1-\varphi')R$ (5.4), and unconditional actuarially fair condition,

$l_3^{\eta}: \varphi' P = (1-\varphi')(1-\eta)R$ (5.5). Furthermore, children's "modified" associated indirect utility including default risk is defined as:

$$\widetilde{\widetilde{v}}^c(P,R,\varphi',\beta,\eta,r,W^c) \equiv \max_{c_1}\begin{pmatrix} \varphi'[u(c_1)+\beta u((1+r)(W_1^c - P - c_1))] \\ +(1-\varphi')(1-\eta)[u(c_1)+\beta u((1+r)(W_0^c + R - c_1))] \\ +(1-\varphi')\eta[u(c_1)+\beta u((1+r)(W_0^c - c_1))] \end{pmatrix}$$

(5.6)

Now I examine the sensitivity of children's optimal demand for an actuarially fair social security $(R^c(\eta), P^c(\eta))$ under (5.4) and (5.5) to η, when η deviates slightly from 0 by a positive bit. Then:

Proposition 5: (Children's demand sensitivity to default risk) Assume that social security has default risk with conditional actuarially fair condition (5.4). Then (i) $R^c{}'(0) < 0, P^c{}'(0) < 0$. Instead of (5.4), assume that social security has default risk with unconditional actuarially fair condition (5.5). Then (ii) $R^c{}'(0) > 0, P^c{}'(0) < 0$.

In either case of (5.4) or (5.5), $R^c(0) = \varphi'(W_1^c - W_0^c)$ and $P^c(0) = (1-\varphi')(W_1^c - W_0^c)$, where children fully insure their income risk.

Sensitivity to taxation

I turn my focus to taxation on social security both for parents and children. Consider two kinds of tax: a lump-sum actuarially fair tax, and an exercise tax only on payment P. Let T_R and T_P be conditional taxes imposed on the realization of receipt R and payment P, respectively. Lump-sum actuarially fair tax is described as (T_R, T_P), where $\varphi T_R = (1-\varphi)T_P$ for parents and $(1-\varphi')T_R = \varphi' T_P$ for children, (5.7). An exercise tax on payment P, is described as $(T_R, T_P) = (0, T_p)$, where $T_P = \zeta P$, and I assume tax-deducted actuarially fair conditions, $l_1^\zeta : \varphi R = (1-\varphi)(1-\zeta)P$ (5.8) for parents, $l_3^\zeta : (1-\varphi')R = \varphi'(1-\zeta)P$ (5.9) for children. ζ is defined as a proportional tax rate on P. The expected tax income by the government is, $ET \equiv (1-\varphi)T_P - \varphi T_R$ (5.10) for parents, and $ET \equiv \varphi' T_P - (1-\varphi')T_R$ (5.11) for children. The associated indirect utilities with regard to (R, P), remain the same as (2.3a) for parents, and (2.3b) for children:

$$\tilde{v}^p(R, P, \varphi, \beta, r, A^p) \equiv u(A^p - P) + \varphi\beta u((1+r)(R+P)) \qquad (2.3a')$$

$$\tilde{v}^c(P, R, \varphi, \beta, r, \overline{W}^c) \equiv \max_{c_1} \begin{pmatrix} \varphi'[u(c_1) + \beta u((1+r)(\overline{W}_1^c - P - c_1))] \\ +(1-\varphi')[u(c_1) + \beta u((1+r)(\overline{W}_0^c + R - c_1))] \end{pmatrix} \qquad (2.3b')$$

Clearly a lump-sum actuarially fair tax is better than an exercise tax on payment P, for both parents and children, in the sense that, keeping the expected tax income ET at constant, a lump-sum actuarially fair tax could always attain better associated indirect utility with regard to (R, P), than an exercise tax only on payment P.

I proceed to the sensitivity analysis to an exercise taxation on payment P, as described in (5.8) for parents and (5.9) for children. (5.8) and (5.9) are, in a sense, equivalent with unconditional actuarially fair conditions incorporating default risk, (5.2) for parents and (5.5) for children, respectively, if I set $(1-\eta)(1-\zeta)=1$. Here, I can interpret η as a conditional profit margin or a subsidy margin on the realization of receipt R. Denote parents' optimal demand for social security with the condition (5.8) by $(\widetilde{R}^p(\zeta), \widetilde{P}^p(\zeta))$, and children's optimal demand for social security with the condition (5.9) by $(\widetilde{R}^c(\zeta), \widetilde{P}^c(\zeta))$, respectively. Then:

Proposition 6: (Parents' demand sensitivity to taxation) Assume that the government imposes an exercise tax on payment P, for parents' social security, with tax-deducted actuarially fair condition (5.8). Then (i) $\widetilde{R}^{p}{}'(\zeta) < 0$, irrespective of the value of σ. Furthermore (ii) $\widetilde{P}^{p}{}'(\zeta) > 0$ if $\sigma > 1$, $\widetilde{P}^{p}{}'(\zeta) = 0$ if $\sigma = 1$, and $\widetilde{P}^{p}{}'(\zeta) < 0$ if $\sigma < 1$.

Proposition 7: (Children's demand sensitivity to taxation) Assume that the government imposes an exercise tax on payment P for children's social security, with tax-deducted actuarially fair condition (5.9). Then (i) $\widetilde{R}^{c}{}'(0) < 0$, irrespective of the value of σ.

Clearly the results shown in this section are quite novel ones, especially as for the sensitivity of parents' indirect (peculiar) utility under mortality risk ((2.3a'), (5.3)). As for that of children's regular indirect utility under regular income risk ((2.3b'), (5.6)), there seem to be some crucially basic analyses with similar settings (Atkinson and Stiglitz (1980, Lecture 4) or Obstfeld and Rogoff (1996)). However I do not still see, to my knowledge, the same results anywhere in the literature, so I dare to leave them here.

6. Application to unfunded (PAYG) social security

At first I explain, using Figure 5, some short-run aspects of intra-family & inter-generational transfer and unfunded social security. $S(R)$ and $B(P)$ axes, beginning from original point O, denote support (receipt) and bequest (payment) between old parents and young adult children under no certainty premium transferred between the generations. a, a', d, d' are indifference curves of old parents' indirect utility regarding insurance contract under their mortality risk, and b, b', e are those of young adult children. The contract curve, which satisfy both Pareto optimality and participation constraints with no actuarially fair insurance available, is GF, and the self-insuring contract curve with no intra-family risk sharing available is DJ. Assume that a fixed level of mandatory certainty premium transfer from young children to old parents, OO'', takes place. Then corresponding to new original point and axes O'', $S'(R')$ and $B'(P')$, DJ (on $\underline{X}_{ac}$) moves to $D'J'$ on line x, and GF ($\underline{X}_p$) does to $G'F'$ on y, respectively. From Lemmas 1 and 2, the slopes of line x and y prove to be less than -1. This aspect also proves Corollary 3 to 7, as well as Proposition 8. For example, corollary 3 to 7 is derived by observing $O''D' > OD$.

Construction of old-age pension scheme

The old-age security scheme can be defined as $\Omega(=\Omega^{ss}+\Omega^{fa})$, where Ω^{ss} and Ω^{fa} represent unfunded social security (public pension) and intra-family & inter-generational (private) transfer, respectively. Ω^{ss} and Ω^{fa} can be divided into two parts, premium (certainty) part $\overline{\Omega}^{ss}$ and $\overline{\Omega}^{fa}$, and actuarially fair risk sharing (insurance) part $\widetilde{\Omega}^{ss}$ and $\widetilde{\Omega}^{fa}$ ($\Omega^{ss} = \overline{\Omega}^{ss} + \widetilde{\Omega}^{ss}$ and $\Omega^{fa} = \overline{\Omega}^{fa} + \widetilde{\Omega}^{fa}$), where $E\widetilde{\Omega}^{ss} = E\widetilde{\Omega}^{fa} = 0$. Assume that unfunded pension tax rate (θ) is imposed on children's disposable income $\overline{W}^{c}$. Then $\overline{\Omega}^{ss} = \theta\overline{W}^{c}$. In Figure 5, it is equal to $OO''(=k)$ (certainty premium), therefore $\overline{\Omega}^{ss} = \theta\overline{W}^{c} = OO'' = k$.

I consider the case in which young adult-hood children implements intra-family bargaining, based on altruism (λ') toward old age parents. Then transfer contract (S, B) is determined so that altruistic indirect utility $\widetilde{v}^{c} + \lambda'\widetilde{v}^{p}$ is maximized. Of course, from two first order conditions with regard to S and B, this altruism motivated contract proves to be Pareto optimal as in competitive case.

Proposition 8: Assume that intra-family bargaining between old-age parents and young adult-hood children is motivated on altruism (λ'), and unfunded pension tax rate (θ) involves only certainty premium transfer $\overline{\Omega}^{ss}$, not actuarially fair insurance $\widetilde{\Omega}^{ss}$, and the participation constraint of each generation regarding intra-family bargaining is not binding. Then, θ does *not* affect overall security scheme $\Omega(= \Omega^{ss} + \Omega^{fa})$.

In other words, Recardian equivalence does hold, as far as taxation financing old age social security is involved in mandatory transfer from neighbor (next), young adult generation. This proposition is presumably equivalent with Proposition 5 in Altig and Davis (1993), but holds only with absence of actuarially fair insurance of social security. Other short-run implications under different assumptions are described as follows.

Corollary 3: Assume again that intra-family bargaining between old-age parents and young adult-hood children is motivated on altruism (λ'), and unfunded pension tax rate (θ)

involves only certainty premium transfer, not actuarially fair insurance, but that the participation constraint of young adult-hood children is now binding. Then the larger θ induces the larger private compensation scheme Ω^{fa} in actuarially fair insurance part $\widetilde{\Omega}^{fa}$ with $Var\widetilde{\Omega}^{fa}(=Var\Omega^{fa})$.

As clear from Figure 5, the only Pareto optimal contract, in which participation constraint for children is binding with no social security available, is represented as point G. If mandatory transfer $\overline{\Omega}^{ss}(=E\Omega^{ss})=OO''$ in certainty premium is imposed, then the only contract moves to G'.

Corollary 4: Assume that intra-family bargaining between old-age parents and young adult-hood children is not available, and unfunded pension tax rate (θ) involves both certainty premium transfer, and fully actuarially fair insurance. Then the larger θ induces the larger insurance scheme Ω^{ss} in both certainty premium $\overline{\Omega}^{ss}$ and actuarially fair insurance $\widetilde{\Omega}^{ss}$ with $Var\widetilde{\Omega}^{ss}(=Var\Omega^{ss})$.

Corollary 5: Assume that intra-family bargaining between old-age parents and young adult-hood children is not available, and unfunded pension tax rate (θ) involves only certainty premium transfer, not actuarially fair insurance. Then the larger θ induces the larger self-insurance cost.

Corollary 6: Assume that intra-family bargaining between old-age parents and young adult-hood adult children is not available. Then one unit of certainty premium transfer from young adulthood generation to old adulthood generation is equivalent with actuarially fair insurance of old-age mortality risk with variance $\varphi/(1-\varphi)$.

As explained in section 3, actuarially fair insurance, which is more than a fixed level in variance, is necessary for unconditionally enabling Arrow-Debreu competitive (state-contingent exchange) economy.

Corollary 7: Assume that intra-family bargaining between old-age parents and young adult-hood children is competitive in Arrow-Debreu fashion, and unfunded pension tax rate (θ) involves both certainty premium transfer $\overline{\Omega}^{ss}$ and a fixed level of actuarially fair insurance $\widetilde{\Omega}^{ss}$. Then the larger θ induces the larger private compensation scheme Ω^{fa} in actuarially fair insurance part $\widetilde{\Omega}^{fa}$ with $Var\widetilde{\Omega}^{fa} (= Var\Omega^{fa})$.

7. Final remarks

In a continuous-time case, parents' problem to be solved can be represented as follows:[11] Define parents' transfer contract incorporating the risk of death $(S(t), B(t))$ for $0 \le t \le T$. T (≒30 years) is the length of stage O, and $(S(t), B(t))$ is measured as a present value at time 0, not t, and $B(t)$ is continuously differentiable for all $0 \le t \le T$. Also let $c(t)$ be their consumption plan at time t, measured as a present value at time t, and A^p be the present value of total available wealth, measured at time 0. Define the probability that parents are alive at time t, as $\varphi(t)$, where $\varphi(0) = 1$, and $\varphi(T) = 0$. Then the budget (feasibility) constraint is written as:

$$\int_0^t \{-S(t') + c(t')\exp(-rt')\}dt' = A^p - B(t) \quad \text{for all } 0 \le t \le T. \quad (7.1)$$

Equivalently in a differential form: $c(t) = [-B'(t) + S(t)]\exp(rt)$ for all $0 \le t \le T$ (7.1')

[11] Basically Yaari (1965) uses the same continuous time representations.

From (7.1), I have $B(0) = A^p$, and $B(T) = 0$. An actuarially fair condition of the transfer contract $(S(t), B(t))$ is: $\int_0^T \varphi(t)S(t)dt = -\int_0^T B(t)\varphi'(t)dt$ (7.2)

Here $\widetilde{S} \equiv \int_0^T \varphi(t)S(t)dt$ is an expected support and $\widetilde{B} \equiv -\int_0^T B(t)\varphi'(t)dt$ is an expected bequest. In a differential form, $\varphi(t)S(t) = -B(t)\varphi'(t)$ (7.2'), although (7.2') is a sufficient condition of (7.2). This is a continuous-time version of an actuarially fair condition in a two-period case, $l_1 : \varphi S = (1-\varphi)B$ (2.2). Assume that parents' transfer schedule $(S(t), B(t))$ is predetermined. Then they solve: (X) $\max_{c(t)} \int_0^T \varphi(t)u(c(t))\exp(-\beta t)dt$ (7.3) s.t. (7.1'). However, as a matter of fact the maximization problem is not left for parents, but their consumption is automatically determined at $c(t) = [-B'(t) + S(t)]\exp(rt)$ for all $0 \le t \le T$ (7.1'). Therefore, as in a two-period case, there exists some possibility of self-insurance, in which parents must pay an additional cost conditionally on death. On the other hand, children's indirect utility at time t with transfer contract $(S(t), B(t))$ ($0 \le t \le T$) proves to have an indifference curve, which is tangent with a continuous-time actuarially fair line $\varphi(t)S(t) = -B(t)\varphi'(t)$ (7.2'). Furthermore these representations on transfer contract $(S(t), B(t))$ prove to be conveniently compatible with description of actuarially fair insurance and self insurance on consumption-based regular utility maximization problems. It also justifies the arguments so far on two-discrete-period transfer (S, B).

From all the above points, my analysis made with two discrete periods does not lose any generality even in a continuous-time case. On the other hand, another direction, which is often adopted in the uncertainty literature, is the approximated transformation of expected utility in the form of risk-premium-incorporated regular (certain) utility. However, this approximation is not necessarily suitable for analysis of life uncertainty, because it tends to miss some distortion effect caused by state dependent nature. Therefore both sides of methodology do result, one because of too much integral complication and the other because of too much ignorance of utility modification, in the failure to capture the essential aspects of intergenerational risk sharing.

The core of this article consists in section 3 and 4, wherein I showed the following points. In the absence of any other insurance mechanism, parents and their children would always want to enter a support-bequest-contract (intergenerational transfers) that would make both risk-averse individuals better off. If the parents also have access to an actuarially fair social security program, however, they would prefer to add intergenerational transfers to that (if social security is rather small) or they would prefer to rely only on their social security contract (if that is sufficiently large). In no event would parents want to rely only on intergenerational transfers if they have an access to social security.[12]

[12] I am indebted to an anonymous referee for this summary.

Another neglected, but crucial issue is the possibility of state contingency in children's future life strategies. For example, if children share parents' mortality risk (fully or partially), their other life strategies (N^c, H^c, A^c) do depend on s_1, their *own* old age mortality risk, as well as other risks (s_2 or s_3), and this might cause two following happenings. Firstly children's indifference curve might not necessarily be tangent with actuarially fair line at a certainty point, and implications stated in this article might be distorted to some extent. Secondly resulting uncertainty might be non-negligibly accumulated over infinite time horizon, and risk-averse children might decrease the expected allocations of N^c, H^c, A^c, as compared with parents' other insurance cases (social security or self insurance). Or the effect might not be so conspicuous especially in case of mutual risk sharing in both life and income uncertainty (s_1 and s_2). Therefore these issues should be separately examined in dynamic context with specific parameter settings.

Thus this article has just shown, using a simple model with two discrete periods and binary states of mortality/income risks, some fundamental propositions regarding the mixture of intergenerational risk sharing and social security. Here for the first time, state-dependent utility under mortality risk proves to generate parents' peculiar indifference curve regarding insurance contract, and self-insurance is shown to play a crucial role on the decision concerning social security holding and intergenerational transfer contract. This article is the first trial to try to capture economic roles of these insurances in a possible mixture, although there are some other papers which consider extreme cases with or without any insurance only. Stated propositions, corollaries and corresponding insights, although described in a simplified manner, are quite novel ones and shed a new light on the literature. Also these results can be easily applied to the dynamic analysis over the long run, for example, in a two-sided altruistic framework, as in Abel (1987) or Altig et al. (1993). As a consequence, this article revisits the issues examined, among many, by Yaari (1965), Abel (1985, 1987), Hurd (1989) and Kotolikoff et al. (1981), and successfully extracts some missing and non-trivial implications in the literature, I believe.

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Y, M, O	Three life stages, for the infant, young and old adulthood
p, c	Two adjacent generations, parents and children (superscripts)
A^p, W^c	Parents' asset holding, and children's disposable income
φ	Probability that old parents are alive at the middle of stage O
φ'	Probability that young adult children earn higher income at the middle of stage M
η	Probability that social security falls in default at the middle of underlying period
β	Constant time preference for each half period
r	Real interest rate for each half period
σ	Constant relative risk aversion coefficient
$u(c)$	Utility of each half period as a function of corresponding consumption
$s_1(=1,0)$	States of parents' being alive or dying, revealed at the middle of stage O
$s_2(=1,0)$	States of children's' earning a higher or lower disposable income alive or dying, revealed at the middle of stage M
$s_3(=1,0)$	States of social security's non-default or default, revealed at the middle of underlying period
(S, B)	Intergenerational transfer contract between parents and children, conditional on $s_1 = 1$ (left) or $s_1 = 0$ (right) $(Support, Bequest)$
(R, P)	Social security contract, conditional on the revelation of s_1, s_2 $(\mathrm{Re}ceipt, Payment)$
c_{M1}, c_{M2}	Consumption of children for the first and second half period during stage M
c_{O1}, c_{O2}	Consumption of parents for the first and second half period during stage O
$\underline{X}_p$	Pareto set of contracts chosen by parents, when only intergenerational transfer with children is possible
$\underline{X}_{ac}$	Pareto set of contracts chosen by parents, when only actuarially fair transfer is possible

Table 1 Main notations and definitions

$\tilde{v}^p(S,B,\varphi,\beta,r,A^p)$	Parents' indirect utility with transfer contract (S,B) available
$\tilde{v}^c(S,B,\varphi,\beta,r,W^c)$	Children's indirect utility with transfer contract (S,B) available
$\tilde{\tilde{v}}^p(R,P,\varphi,\eta,\beta,r,A^p)$	Parents' indirect utility with under-default-risk transfer contract (R,P) available
$\tilde{\tilde{v}}^c(P,R,\varphi',\eta,\beta,r,W^c)$	Children's indirect utility with under-default-risk transfer contract (R,P) available
$V^p(K)$, $V^p(Y)$, $V^p(K+Y)$	Parents' maximized utilities, attained by concluding only social security K, only intergenerational transfer Y, and both of them
$E^\eta:(R_E(\eta),P_E(\eta))$	Optimal point of parents' default-risk-incorporated indirect utility $\tilde{\tilde{v}}^p$ (5.3) on l_1 (5.1) or l_1^η (5.2)
$I^\eta:(R_I(\eta),P_I(\eta))$	Minimum utility point of parents' default-risk-incorporated indirect utility $\tilde{\tilde{v}}^p$ (5.3) on l_1 (5.1) or l_1^η (5.2)
$(R^c(\eta),P^c(\eta))$	Optimal point of children's default-risk-incorporated indirect utility $\tilde{\tilde{v}}^c$ (5.6) on l_3 (5.4) or l_3^η (5.5)
$(\tilde{R}^p(\zeta),\tilde{P}^p(\zeta))$	Optimal point of parents' indirect utility $\tilde{v}^p$ (2.3a') on l_1^ζ (5.8)
$(\tilde{R}^c(\zeta),\tilde{P}^c(\zeta))$	Optimal point of children's indirect utility $\tilde{v}^c$ (2.3b') on l_3^ζ (5.9)
$\Omega^{ss},\Omega^{fa},\Omega(=\Omega^{ss}+\Omega^{fa})$	Unfunded social security (public pension), intra-family & inter-generational (private) transfer, overall old-age security
$\overline{\Omega}^{ss},\overline{\Omega}^{fa},\overline{\Omega}$	Certainty premium part of $\Omega^{ss},\Omega^{fa},\Omega$
$\tilde{\Omega}^{ss},\tilde{\Omega}^{fa},\tilde{\Omega}$	Actuarially fair insurance part of Ω^{ss}, Ω^{fa}, Ω ($E\tilde{\Omega}^{ss}=0$, $E\tilde{\Omega}^{fa}=0$, $E\tilde{\Omega}=0$)

Table 1 Main notations and definitions (Continued)

(2.2)	$l_1 : \varphi S = (1-\varphi)B$ or $l_1 : \varphi R = (1-\varphi)P$: Actuarially fair condition
(2.3a), (2.3a')	$\tilde{v}^p(S,B,\varphi,\beta,r,A^p) \equiv u(A^p - B) + \varphi\beta u((1+r)(S+B))$
(2.3b), (2.3b')	$\tilde{v}^c(S,B,\varphi,\beta,r,W^c) \equiv \max_{c_1}\begin{pmatrix}\varphi[u(c_{M1}) + \beta u((1+r)(W^c - S - c_{M1}))] \\ +(1-\varphi)[u(c_{M1}) + \beta u((1+r)(W^c + B - c_{M1}))]\end{pmatrix}$
(5.3)	$\tilde{\tilde{v}}^p(R,P,\varphi,\eta,\beta,r,A^p)$ $\equiv u(A^p - P) + \varphi(1-\eta)\beta u((1+r)(R+P)) + \varphi\eta\beta u((1+r)P)$
(5.6)	$\tilde{\tilde{v}}^c(P,R,\varphi',\beta,\eta,r,W^c)$ $\equiv \max_{c_1}\begin{pmatrix}\varphi'[u(c_1) + \beta u((1+r)(W_1^c - P - c_1))] \\ +(1-\varphi')(1-\eta)[u(c_1) + \beta u((1+r)(W_0^c + R - c_1))] \\ +(1-\varphi')\eta[u(c_1) + \beta u((1+r)(W_0^c - c_1))]\end{pmatrix}$
	$l_1' : (1-\varphi)B - \varphi S = k$: Constant premium line
	$l_1'' : (1-\varphi)B - \varphi S = k'$: Constant premium line passing point O'
	$l_2 : B = -S$: Certainty line
(5.1)	$l_1 : \varphi R = (1-\varphi)P$: Parents' conditional actuarially fair condition on non-default
(5.2)	$l_1^\eta : \varphi(1-\eta)R = (1-\varphi)P$: Parents' unconditional actuarially fair condition
(5.4)	$l_3 : \varphi' P = (1-\varphi')R$: Children's conditional actuarially fair condition on non-default
(5.5)	$l_3^\eta : \varphi' P = (1-\varphi')(1-\eta)R$: Children's unconditional actuarially fair condition
(5.8)	$l_1^\zeta : \varphi R = (1-\varphi)(1-\zeta)P$: Parents' tax-deducted actuarially fair condition
(5.9)	$l_3^\zeta : (1-\varphi')R = \varphi'(1-\zeta)P$: Children's tax-deducted actuarially fair condition

Table 2 Main equations and definitions

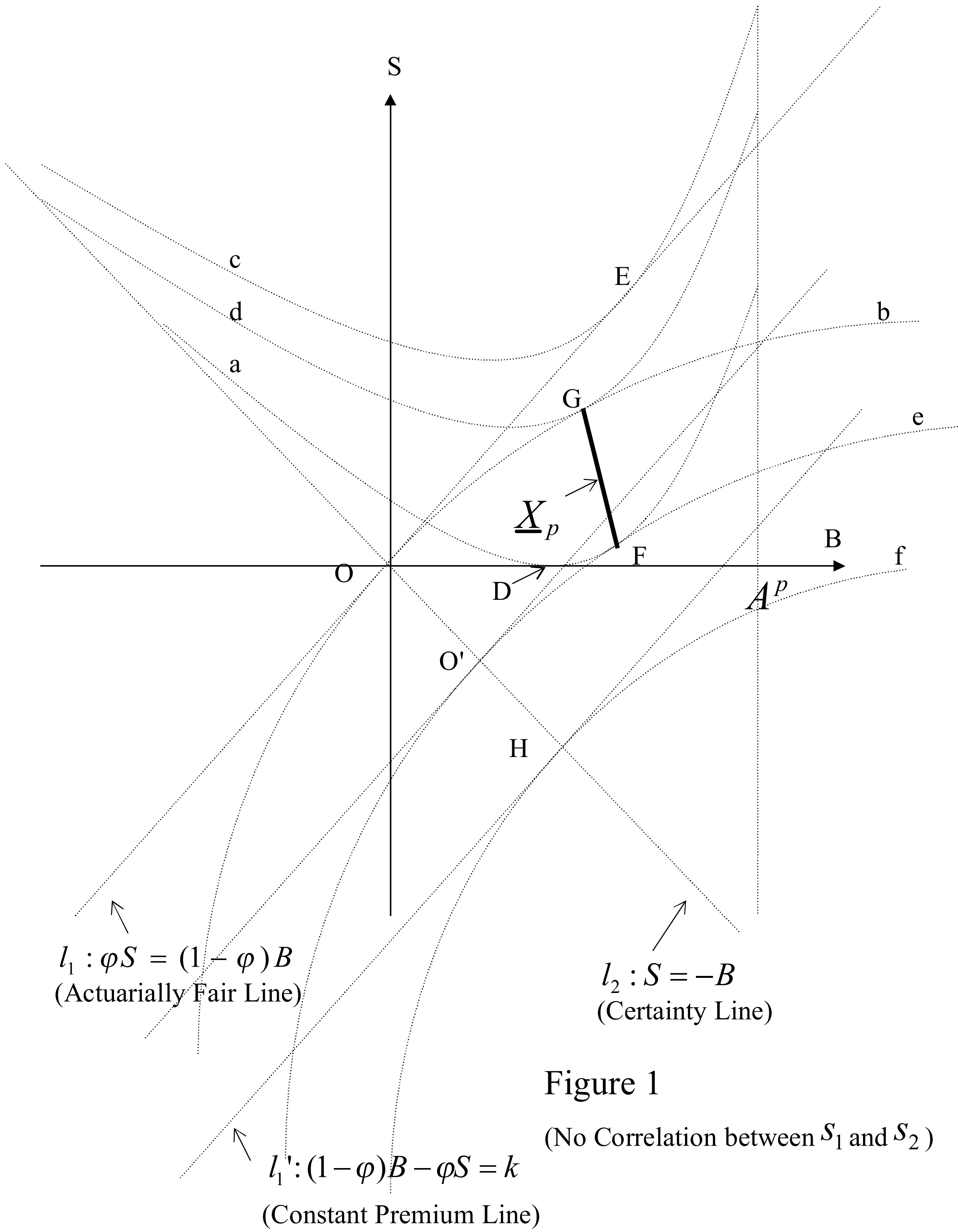

Figure 1

(No Correlation between S_1 and S_2)

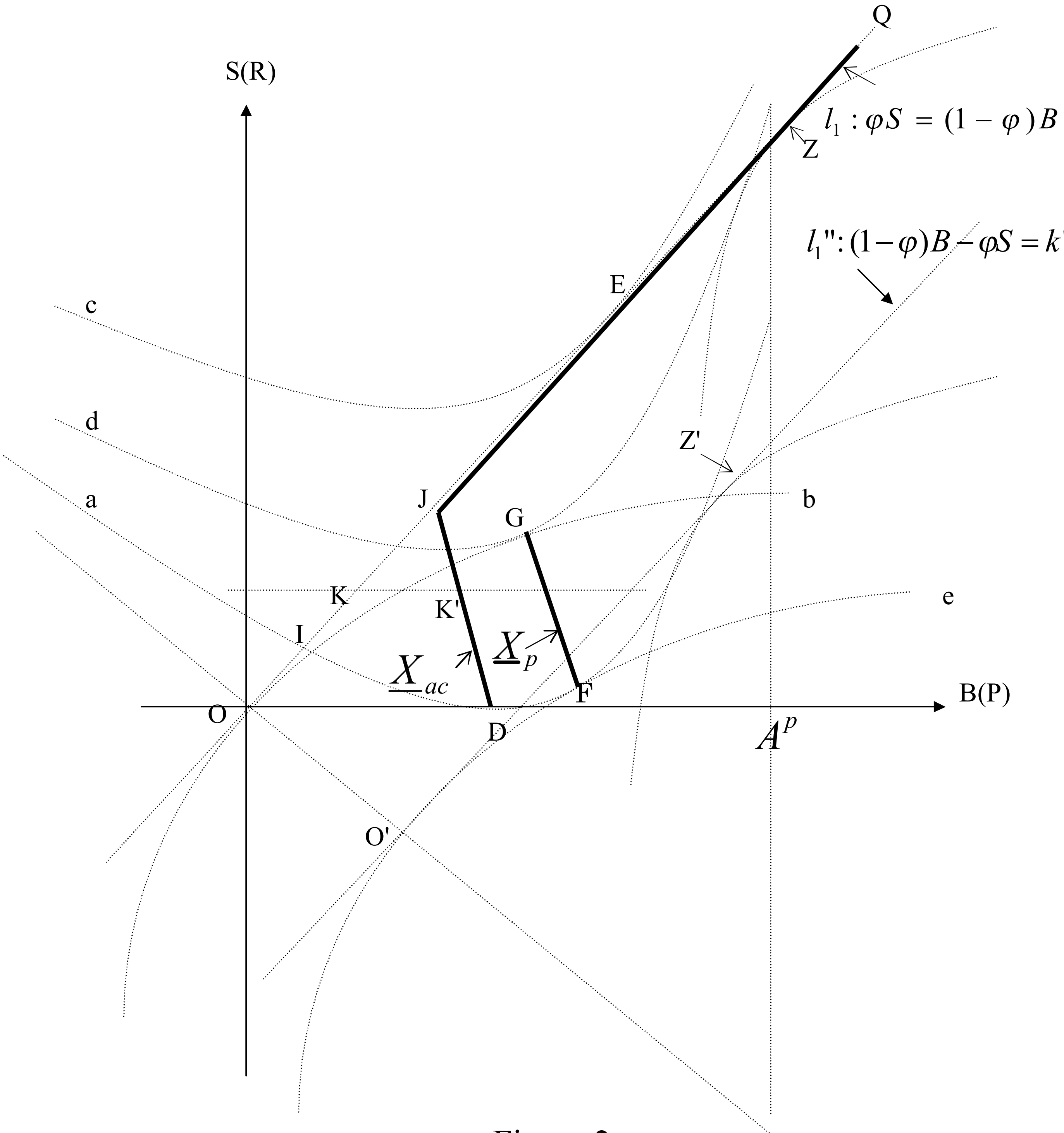

Figure 2

(No correlation between s_1 and s_2) $l_2 : S = -B$

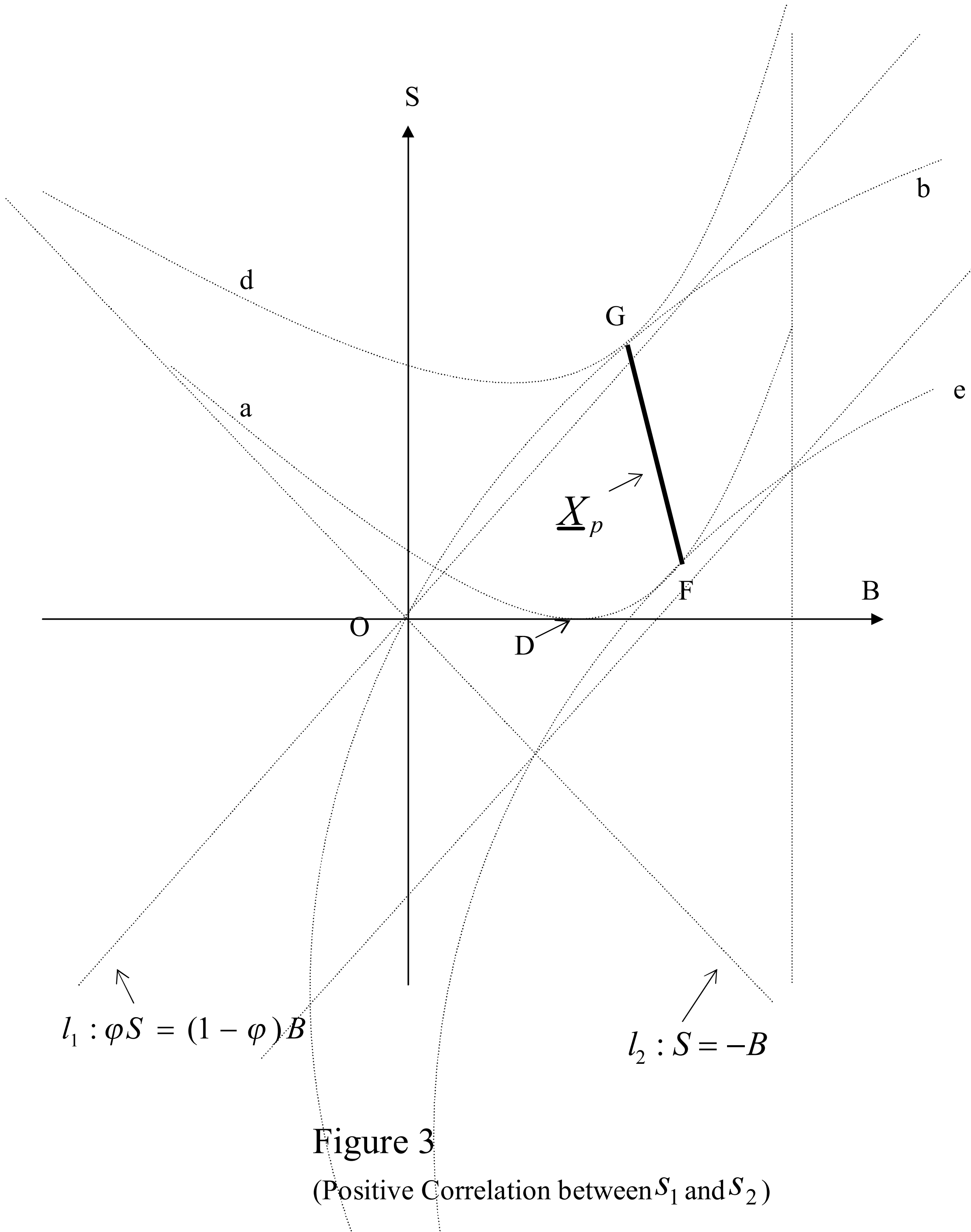

Figure 3

(Positive Correlation between s_1 and s_2)

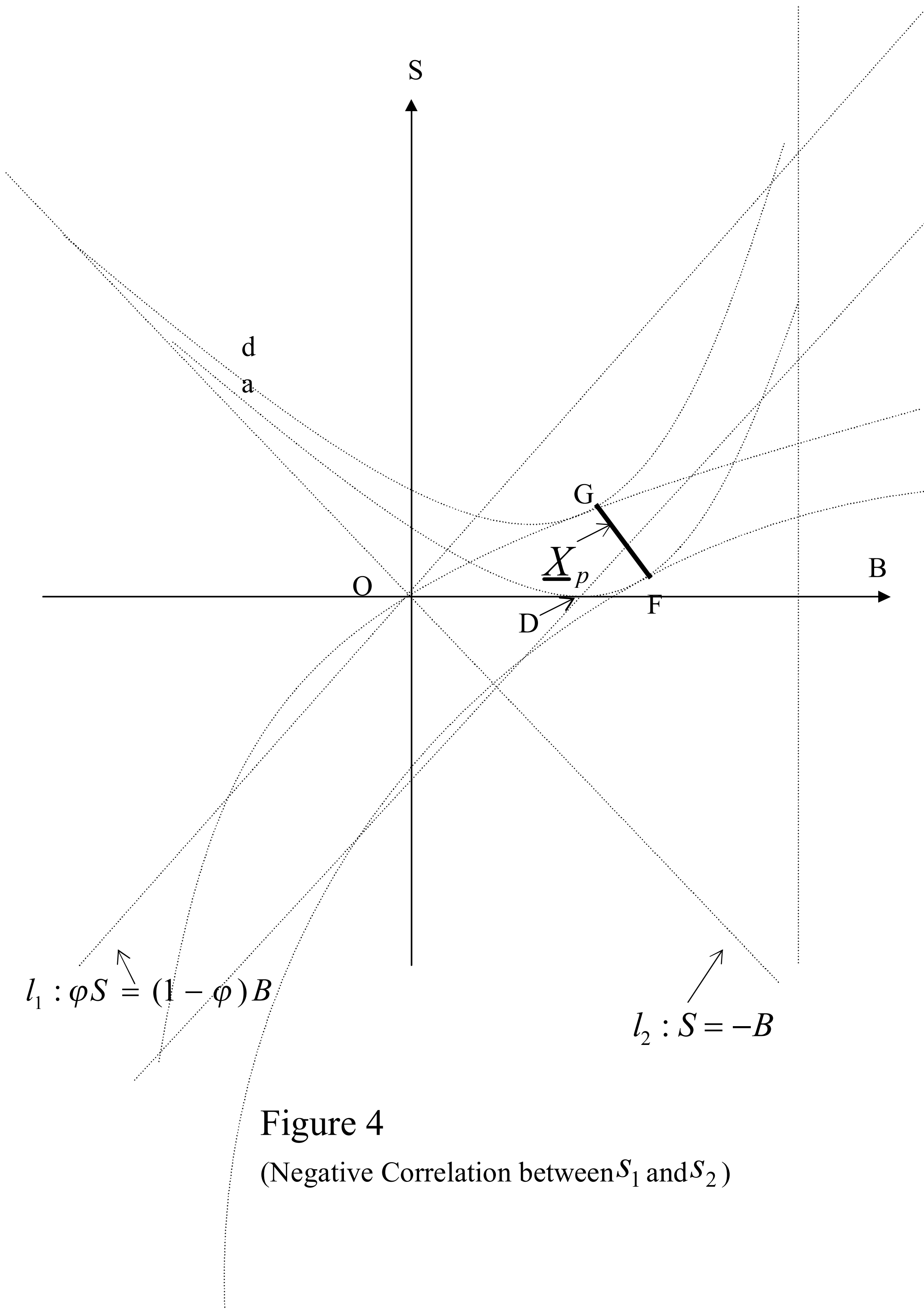

Figure 4

(Negative Correlation between S_1 and S_2)

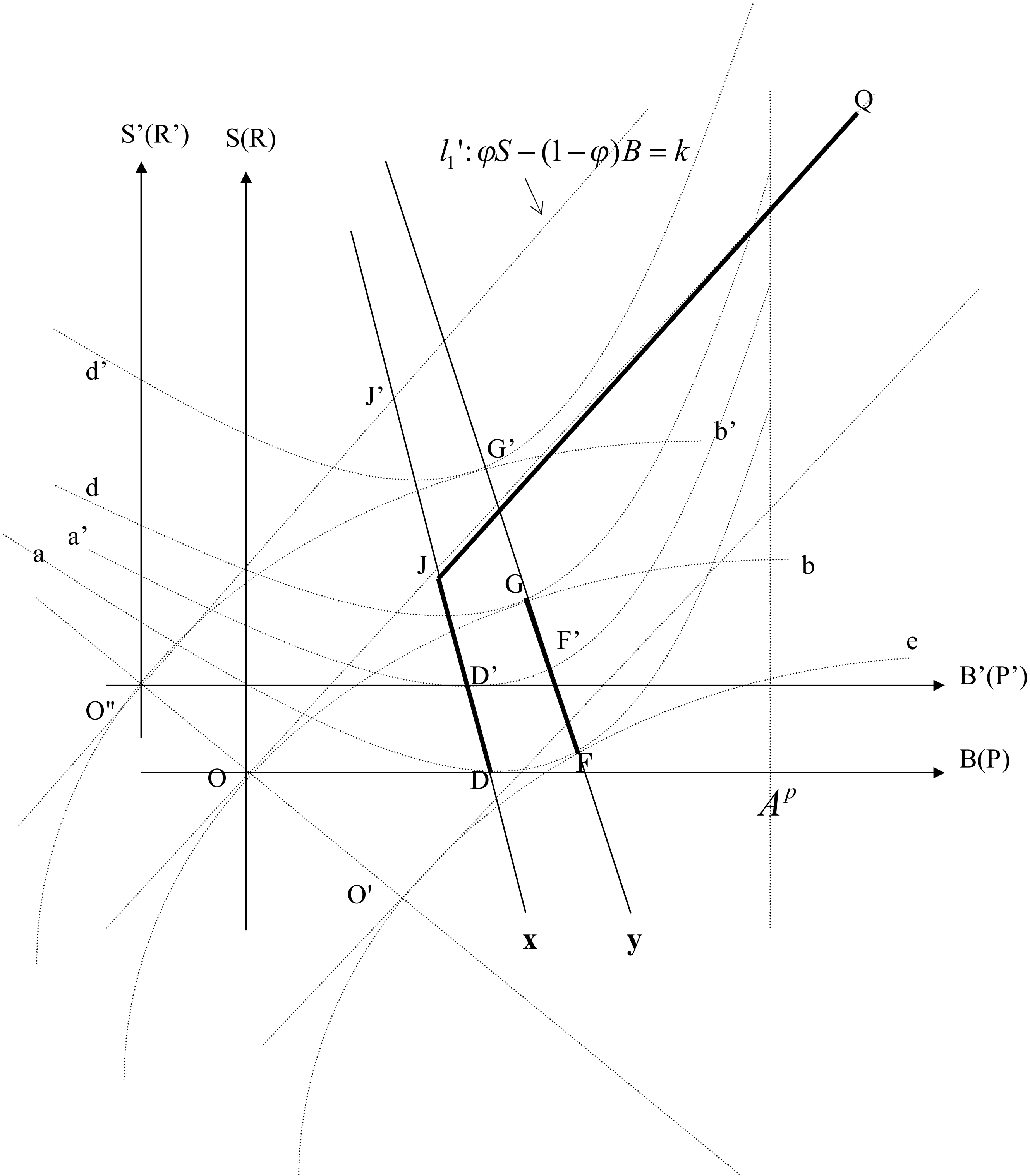

Figure 5

Intra-family & Inter-generational Transfer and Unfunded Social Security

$l_2 : S = -B$

Appendix 1 (Proofs in section 4)

Rigorous definitions of $V^P(K)$, $V^P(Y)$ *and* $V^P(K+Y)$

Define an actuarially fair social security $K:(R_K, P_K)$, where K is on $l_1: \varphi R_K = (1-\varphi)P_K$, and an intergenerational transfer $Y:(S_Y, B_Y)$ on $\underline{X}_P$. Then:

$$V^p(K) \equiv \max_{P'\geq 0} \tilde{v}^p(R_K, P_K + P', \varphi, \beta, r, A^p) \quad \text{(A1.1-1)}$$ [13]

$$V^p(Y) \equiv \tilde{v}^p(S_Y, B_Y, \varphi, \beta, r, A^p) \quad \text{(A1.1-2)}$$

$$V^p(K+Y) \equiv \max_{(S,B),P'\geq 0} \tilde{v}^p(R_K + S, P_K + B + P', \varphi, \beta, r, A^p) \quad \text{(A1.1-3)}$$ [14]

$$\text{s.t. } \tilde{v}^c(S, B, \varphi, \beta, r, \overline{W}^c) = \tilde{v}^c(S_Y, B_Y, \varphi, \beta, r, \overline{W}^c)(\equiv \overline{v}^c)$$

$$\text{and } l_1: \varphi R_K = (1-\varphi)P_K$$

<u>Proof of Proposition 1</u>:

See Figure 2. Graphically $V^P(K)$ can be determined as parents' indirect utility of the point, where a horizontal line $R = R_K$, crosses $\underline{X}_{ac}$ (that is, parents' indifference curve is tangent with a horizontal line $R = R_K$ clearly on $\underline{X}_{ac}$), $V^P(Y)$ simply as that of point Y

[13] (A1.1-1) can be rewritten as:

$$V^p(K) \equiv \max_{R=R_K, P\geq P_K} \tilde{v}^p(R, P, \varphi, \beta, r, A^p) \quad \text{(A1.1-1')}$$

[14] (A1.1-3) can be rewritten as:

$$V^p(K+Y) \equiv \max_{(R,P),P'\geq 0} \tilde{v}^p(R, P + P', \varphi, \beta, r, A^p) \quad \text{(A1.1-3')}$$

$$\text{s.t. } \tilde{v}^c(R - R_K, P - P_K, \varphi, \beta, r, \overline{W}^c) = \tilde{v}^c(S_Y, B_Y, \varphi, \beta, r, \overline{W}^c)(\equiv \overline{v}^c)$$

$$\text{and } l_1: \varphi R_K = (1-\varphi)P_K$$

on $\underline{X}_P$, and $V^P(K+Y)$ as that of the point where parents' indifference curve is tangent with the curve generated by shifting children's indifference curve in parallel along with l_1 by vector (R_K, P_K).

(i) (Case: $Y = G$) Let Z be the point where children's "shifted" indifference curve, which is tangent both with parents' indifference curve d, and an actuarially fair line l_1, is tangent with l_1. For some social security K, children's shifted indifference curve is denoted by b' for the original indifference curve b. Graphically it is clear that for any arbitrary K on l_1 such that $OK \le OZ$, parents' indifference curve, which is tangent with children's shifted indifference curve b', is located in the upper side of both point K and indifference curve d (or point G). So, I have $V^p(Y(=G)) \le V^p(K+Y(=G))$ with equality when $K = Z$, and $V^p(K) \le V^p(K+Y(=G))$ with equality when $K = E$.

(ii) (Case: $Y = F$) Denote the point where children's indifference curve e crosses the certainty line $l_2 : R = -P$, by O'. Also denote some constant premium line, which passes O', by $l_1'': (1-\varphi)P - \varphi R = k'$. (So, e is tangent with parents' indifference curve a at point F, and also is tangent with l_1'' at point O'.) Z' is the point where children's shifted indifference curve, which is tangent with both parents' indifference curve a and a constant premium line l_1'', is tangent with l_1''. For some social security K, denote children's shifted indifference curve, which corresponds with original indifference curve e, by e'. Graphically it is clear that for any arbitrary K on l_1, such that $OK \leq O'Z'$, that parents' indifference curve, which is tangent with children's shifted indifference curve e', is located in the upper side of parents' indifference curve a (or point F). So I have $V^p(Y(=F)) \leq V^p(K+Y(=F))$ with equality when $OK = O'Z'$. Now I proceed to the proof of $V^p(K) \geq V^p(K+Y(=F))$. For later convenience, I rewrite $V^p(K)$ and $V^p(K+Y)$ as $V^p(R_K)$ and $V^p(R_K, Y)$ using the amount of receipt for social security K, R_K. Clearly, when $K = O$ (zero receipt, zero payment), I have $V^p(R_K(=0)) = V^p(R_K(=0), Y(=F))$,[15] because parents are indifferent between D and F. So, at first I show that,

$$\frac{\partial}{\partial R_K} V^p(R_K(=0)) > \frac{\partial}{\partial R_K} V^p(R_K(=0), Y(=F)), \tag{A1.2}$$

and next show that $V^p(R_K, Y(=F))$ can never catch and overpass $V^p(R_K)$ for any arbitrary K such that $OK \leq O'Z'$.

First step: Since, given an intergenerational transfer Y, parents need not pay an additional cost on death (i.e., one constraint $P' \geq 0$ *is* binding.), I can rewrite (A1.1-3) as:

$$V^p(R_K, Y) \equiv \max_{(R,P)} \tilde{v}^p(R, P, \cdots) \tag{A1.3}$$

$$\text{s.t. } \tilde{v}^c(R - R_K, P - P_K, \cdots) = \tilde{v}^c(S_Y, B_Y, \cdots)(\equiv \bar{v}^c)$$

$$\text{and } l_1 : \varphi R_K = (1-\varphi) P_K$$

[15] That is, $V^p(K(=O)) = V^p(K(=O) + Y(=F))$.

Here (R,P) denotes a "mixed" transfer contract schedule. Plugging the second equation of constraints into other equations of (A1.3), I have a following Lagrangian and corresponding first order conditions.

$$
\begin{aligned}
&L = \tilde{v}^{p}(R,P,\cdots) - \lambda\left(\tilde{v}^{c}(R-R_K, P-\frac{\varphi}{1-\varphi}R_K,\cdots) - \bar{v}^{c}\right)\\
&\frac{\partial}{\partial R}\tilde{v}^{p}(R,P,\cdots) - \lambda\frac{\partial}{\partial R}\tilde{v}^{c}(R-R_K, P-\frac{\varphi}{1-\varphi}R_K,\cdots) = 0\\
&\frac{\partial}{\partial P}\tilde{v}^{p}(R,P,\cdots) - \lambda\frac{\partial}{\partial P}\tilde{v}^{c}(R-R_K, P-\frac{\varphi}{1-\varphi}R_K,\cdots) = 0
\end{aligned}
\tag{A1.4}
$$

From the envelope theorem, also using (A1.3), I obtain:

$$
\begin{aligned}
&V^{p}{}'(R_K,Y) \equiv \frac{\partial}{\partial R_K}V^{p}(R_K,Y)\\
&= \lambda\left(\frac{\partial}{\partial R}\tilde{v}^{c}(R-R_K, P-\frac{\varphi}{1-\varphi}R_K,\cdots) + \frac{\varphi}{1-\varphi}\frac{\partial}{\partial P}\tilde{v}^{c}(R-R_K, P-\frac{\varphi}{1-\varphi}R_K,\cdots)\right)\\
&= \frac{\partial}{\partial R}\tilde{v}^{P}(R,P,\cdots) + \frac{\varphi}{1-\varphi}\frac{\partial}{\partial P}\tilde{v}^{P}(R,P,\cdots) \equiv \Xi(R,P)
\end{aligned}
\tag{A1.5}
$$

Similarly I obtain the following equation quite easily:

$$
\begin{aligned}
&V^{p}{}'(R_K) \equiv \frac{\partial}{\partial R_K}V^{p}(R_K)\\
&= \frac{\partial}{\partial R}\tilde{v}^{P}(R,P,\cdots)
\end{aligned}
\tag{A1.6}
$$

Denote the point at which parents indifference curve, which crosses a solution point of (A1.3), $\Omega:(R_\Omega, P_\Omega)$ say, intersects with $\underline{X}_{ac}$, by $\Omega_{ac}:(R_{\Omega_{ac}}, P_{\Omega_{ac}})$. Clearly, at any arbitrary point on $\underline{X}_{ac}$, I have $\frac{\partial}{\partial P}\tilde{v}^{P}(R,P,\cdots)=0$, so (A1.5) and (A1.6) actually share the same value at Ω and Ω_{ac}, that is $\Xi(R,P)$. Now I have only to show that $\Xi(R,P)$ is decreasing as P increases (moves) along parents' indifference curve from Ω_{ac} to Ω.

Here, denote parents' indifference curve, which passes Ω_{ac} and Ω, by g. Remember parents' indirect utility:

$$\tilde{v}^{p}(R,P,\varphi,\beta,r,A^{p}) \equiv u(A^{p}-P)+\varphi\beta u((1+r)(R+P)) \tag{2.3a}$$

For the first order condition, I have $\frac{\partial}{\partial R}\tilde{v}^{P}(R,P,\cdots)>0$. Furthermore, if P is located in the right side of DJ on $\underline{X}_{ac}$), then $\frac{\partial}{\partial P}\tilde{v}^{P}(R,P,\cdots)<0$

For the second order condition, it follows directly that:

$$\begin{aligned}
&\frac{\partial^2}{\partial R^2}\tilde{v}^{P}(R,P,\cdots)=\frac{\partial^2}{\partial P\partial R}\tilde{v}^{P}(R,P,\cdots)<0\\
&\frac{\partial^2}{\partial P^2}\tilde{v}^{P}(R,P,\cdots)<0
\end{aligned} \tag{A1.7}$$

From the definition, I also have;

$$\Xi(R,P)\equiv\frac{\partial}{\partial R}\tilde{v}^{P}(R,P,\cdots)+\frac{\varphi}{1-\varphi}\frac{\partial}{\partial P}\tilde{v}^{P}(R,P,\cdots) \tag{A1.8}$$

So, from (A1.7) and (A1.8), it follows that:

$$\begin{aligned}
&\frac{\partial}{\partial R}\Xi(R,P)\equiv\frac{\partial^2}{\partial R^2}\tilde{v}^{P}(R,P,\cdots)+\frac{\varphi}{1-\varphi}\frac{\partial^2}{\partial P\partial R}\tilde{v}^{P}(R,P,\cdots)<0\\
&\frac{\partial}{\partial P}\Xi(R,P)\equiv\frac{\partial^2}{\partial P\partial R}\tilde{v}^{P}(R,P,\cdots)+\frac{\varphi}{1-\varphi}\frac{\partial^2}{\partial P^2}\tilde{v}^{P}(R,P,\cdots)<0
\end{aligned} \tag{A1.9}$$

$\frac{dR}{dP}|_{*}>0$ (As graphically clear, P increases (moves) along parents indifference curve from Ω_{ac} to Ω, R also increases.)

where * denotes that (R,P) is on $g:\tilde{v}^{p}(R,P,\cdots)=\bar{v}^{p}_{\Omega,\Omega_{ac}}$, say.

Now, since $\frac{d}{dP}\Xi(R,P)|_{*}=\frac{\partial}{\partial R}\Xi(R,P)\frac{dR}{dP}|_{*}+\frac{\partial}{\partial P}\Xi(R,P)$, (A1.10)

and also $dP>0$, from (A1.9) and (A1.10) I get $\frac{d}{dP}\Xi(R,P)|_{*}<0$ (A1.11). If $\Omega=F$, clearly $\Omega_{ac} = D$, so I have completed the first step, that is, have proved (A1.2).

Second step: Assume that for some $K:(R_K, P_K)$, I have $V^p(K) = V^p(K + Y(=F))$, in another expression, $V^p(R_K) = V^p(R_K, Y(=F))$. For this K, denote solution points (of mixed contract schedule (R,P)) for (A1.1-3') and (A1.3), by Ω again, and $\Sigma:(R_\Sigma, P_\Sigma)$, respectively. Ω and Σ attain the same indirect utility for parents, so Σ should coincide with Ω_{ac} in the above notation. Then I can use the same inequality (A1.11), in order to prove:

$$\frac{\partial}{\partial R_K} V^p(R_K(=R_\Sigma = R_{\Omega_{ac}})) > \frac{\partial}{\partial R_K} V^p(R_K(=R_\Omega), Y(=F)) \qquad \text{(A1.12)}$$

From the continuity of $V^p(R_K)$ and $V^p(R_K, Y(=F))$ with regard to R_K, now I have just proved $V^p(R_K) \geq V^p(R_K, Y(=F))$, that is, $V^p(K) \geq V^p(K + Y(=F))$ for any arbitrary K such that $OK \leq O'Z'$.

<u>Proof of Lemma 1:</u>

Graphically it is clear that, for any points on a segment line JD of $\underline{X}_{ac}$, which is the subset of solution points for (A1.1-1), one constraint $P' \geq 0$ is *not* binding. So, from the first order condition, I have a following equality:

$$\frac{\partial \tilde{v}^p}{\partial P} = -u'(A^p - P) + \varphi\beta(1+r)u'\big((1+r)(R_K + P)\big) = 0$$

(A2.1)

Taking R as a function of P, and differentiate (A2.1) with regard to P, I obtain:

$$u''(A^p - P) + \varphi\beta(1+r)^2 u''\big((1+r)(R_K + P)\big)\left(\frac{dR_K}{dP} + 1\right) = 0 \qquad \text{(A2.2)}$$

Considering $u'' < 0$, it follows directly that $\frac{dR_K}{dP} < -1$ (A2.3).

Proof of Lemma 2:

Plugging the constant premium condition $l_1' = (1-\varphi)P - \varphi R = k$ (A2.4) into (2.3a), I obtain the first order condition:

$$-u'(A^p - P) + \beta(1+r)u'\left((1+r)\left(\frac{P-k}{\varphi}\right)\right) = 0 \qquad \text{(A2.5)}$$

Taking P as a function of k, and differentiate (A2.4) with regard to k, I obtain;

$$u''(A^p - P)\frac{dP}{dk} + \frac{\beta(1+r)^2}{\varphi}u''\left((1+r)\left(\frac{P-k}{\varphi}\right)\right)\left(\frac{dP}{dk} - 1\right) = 0\ , \qquad \text{(A2.6)}$$

from which it follows that $0 < \frac{dP}{dk} < 1$ or $\frac{dk}{dP} > 1$. Since, from (A2.4);

$$\frac{dR}{dP} = \frac{1-\varphi}{\varphi} - \frac{1}{\varphi}\frac{dk}{dP}, \qquad \text{(A2.7)}$$

I have $\frac{dR}{dP} < -1$. Now the proof is done.

Appendix 2 (Proofs in section 5)

Proof of Proposition 4:

Proof of (i): Parents' associated indirect utility including default risk is given in (5.3). Maximizing (5.3) subject to (5.1) with regard to P (and implicitly R), I have the first order condition for point E^{η}:

$$-u'(A^p - P) + (1-\eta)\beta(1+r)u'((1+r)P/\varphi) + \varphi\eta\beta(1+r)u'((1+r)P) = 0 \quad \text{(A3.1)}$$

Replacing P with $P(\eta)$ as a function of η in (A3.1), differentiating the equality with regard to η, and implementing the comparative statics immediately produces the following equation:

$$\begin{aligned} &P'(\eta)\{u''(A^p - P(\eta)) + [(1-\eta)\beta(1+r)^2/\varphi]u''((1+r)P(\eta)/\varphi) \\ &+ \varphi\eta\beta(1+r)^2 u''((1+r)P(\eta))\} \\ &= \beta(1+r)[u'((1+r)P(\eta)/\varphi) - \varphi u'((1+r)P(\eta))] \end{aligned} \quad \text{(A3.2)}$$

The coefficient of $P'(\eta)$ in L.H.S,

$$\{u''(A^p - P(\eta)) + [(1-\eta)\beta(1+r)^2/\varphi]u''((1+r)P(\eta)/\varphi) + \varphi\eta\beta(1+r)^2 u''((1+r)P(\eta))\},$$

is clearly negative since $u'' < 0$. The sign of R.H.S is negative, zero and positive, corresponding to $\sigma > 1$, $\sigma = 1$ and $\sigma < 1$, respectively. Replacing $P(\eta)$ with $P_E(\eta)$, I have completed the proof.

Proof of (iii): The proof is almost the same as proof of (i) except for maximizing (5.3) subject to (5.2) instead of (5.1). The first order condition for point E^{η} is:

$$\begin{aligned} &-u'(A^p - P(\eta)) + \beta(1+r)[(1-\varphi) + \varphi(1-\eta)]u'\big((1+r)P(\eta)[(1-\varphi) + \varphi(1-\eta)]/(\varphi(1-\eta))\big) \\ &+ \varphi\eta\beta(1+r)u'((1+r)P(\eta)) = 0 \end{aligned}$$

(A3.3)

I get a following equation for comparative statics:

$$
\begin{aligned}
&P'(\eta)\{u''(A^p - P(\eta)) \\
&+\beta(1+r)^2([(1-\varphi)+\varphi(1-\eta)]^2/(\varphi(1-\eta)))u''\big((1+r)P(\eta)[(1-\varphi)+\varphi(1-\eta)]/(\varphi(1-\eta))\big) \\
&+\varphi\eta\beta(1+r)^2u''((1+r)P(\eta))\} \\
&=\beta(1+r)^2[(1-\varphi)+\varphi(1-\eta)]((1-\varphi)/(\varphi(1-\eta)^2))u''\big((1+r)P(\eta)[(1-\varphi)+\varphi(1-\eta)]/(\varphi(1-\eta))\big) \\
&-\varphi\beta(1+r)u'((1+r)P(\eta))
\end{aligned}
$$

(A3.4)

Considering $u'>0$, $u''<0$, clearly, R.H.S is negative, and the coefficient of $P'(\eta)$ in L.H.S,

$$
\begin{aligned}
&\{u''(A^p - P(\eta)) \\
&+\beta(1+r)^2[(1-\varphi)+\varphi(1-\eta)]^2/(\varphi(1-\eta))]u''\big((1+r)P(\eta)[(1-\varphi)+\varphi(1-\eta)]/(\varphi(1-\eta))\big) \\
&+\varphi\eta\beta(1+r)^2u''((1+r)P(\eta))\}
\end{aligned}
$$

is also negative. So, replacing $P(\eta)$ with $P_E(\eta)$, the proof is done.

Proof of (ii): Since (5.3) must be constant subject to (5.1), I have:

$$u(A^p - P(\eta)) + \varphi(1-\eta)\beta u((1+r)P(\eta)/\varphi) + \varphi\eta\beta u((1+r)P(\eta)) = Const.\ \text{over}\ \eta$$

The first order condition with regard to η produces:

$$
\begin{aligned}
&P'(\eta)\{-u'(A^p - P(\eta)) + (1-\eta)\beta(1+r)u'((1+r)P(\eta)/\varphi) \\
&+\varphi\eta\beta(1+r)u'((1+r)P(\eta))\} \qquad \text{(A3.5)} \\
&=\varphi\beta[u((1+r)P(\eta)/\varphi) - u((1+r)P(\eta))] > 0
\end{aligned}
$$

Evaluating (A3.5) at $\eta = 0$,

$$P'(0)\{-u'(A^p - P(0)) + \beta(1+r)u'((1+r)P(0)/\varphi)\} > 0 \qquad \text{(A3.6)}$$

So, I need the sign of the coefficient of $P'(0)$ in L.H.S,

$$\{-u'(A^p - P(0)) + \beta(1+r)u'((1+r)P(0)/\varphi)\}. \qquad \text{(A3.7)}$$

But, this is exactly the first order condition at $E^{\eta=0}(=E)$, which should be 0 at E. Since

$I^{\eta=0}(=I)$ is located in the left-down side of E along l_1, so (A3.7) should have a positive value. The proof is now done.

Proof of (iv): Since (5.3) must be constant subject to (5.2), I have:

$$\begin{aligned} &u(A^p - P(\eta)) + \varphi(1-\eta)\beta u\big((1+r)P(\eta)[(1-\varphi)+\varphi(1-\eta)]/(\varphi\beta(1-\eta))\big) \\ &+\varphi\eta\beta u((1+r)P(\eta)) = Const. \end{aligned}$$

over η. (A3.8)

Differentiating (A3.8) with regard to η produces:

$$\begin{aligned} &P'(\eta)\{-u'(A^p - P(\eta)) \\ &+\beta(1+r)[(1-\varphi)+\varphi(1-\eta)]u'\big((1+r)P(\eta)[(1-\varphi)+\varphi(1-\eta)]/(\varphi(1-\eta))\big) \\ &+\varphi\eta\beta(1+r)u'((1+r)P(\eta))\} \\ &=\varphi\beta[u\big((1+r)P(\eta)[(1-\varphi)+\varphi(1-\eta)]/(\varphi(1-\eta))\big) - u((1+r)P(\eta))] \\ &+\beta(1+r)(1-\varphi)/(1-\eta)u'\big((1+r)P(\eta)[(1-\varphi)+\varphi(1-\eta)]/(\varphi(1-\eta))\big)P(\eta) \end{aligned}$$

(A3.9)

Evaluating (A3.9) at $\eta = 0$,

$$\begin{aligned} &P'(0)\{-u'(A^p - P(0)) + \beta(1+r)u'((1+r)P(0)/\varphi)\} \\ &=\varphi\beta[u((1+r)P(0)/\varphi) - u((1+r)P(0))] \\ &+\beta(1+r)(1-\varphi)u'((1+r)P(0)/\varphi)P(0) \\ &>0 \end{aligned} \qquad \text{(A3.10)}$$

Now I have only to examine the sign of the coefficient of $P'(0)$ in L.H.S;

$\{-u'(A^p - P(0)) + \beta(1+r)u'((1+r)P(0)/\varphi)\}$, which is exactly the same as (A3.7). Since $l_1^{\eta=0} = l_1$, I can apply the same argument as after (A3.7) in the proof of (ii). The proof is now done.

Proof of Proposition 5:

Proof of (i): Children's associated indirect utility including default risk is given in (5.6). Maximizing (5.6) subject to (5.4) with regard to R (and implicitly P) and c_1, I have the first order condition, with regard to R, for some value of c_1 such that $\hat{c}_1 = \arg\max_{c_1,P,R} \tilde{\tilde{v}}^c(P,R,\varphi',\beta,\eta,r,\overline{W}^c)$ s.t. (5.4):

$$1-\eta = \frac{u'\left((1+r)(\overline{W}_1^c - \frac{(1-\varphi')}{\varphi'}R - \hat{c}_1)\right)}{u'\left((1+r)(\overline{W}_0^c + R - \hat{c}_1)\right)} \quad \text{(A3.11)}$$

Since $u'()$ is a decreasing function ($u''<0$), for $\eta > 0$ I have:

$$\overline{W}_1^c - \frac{1-\varphi'}{\varphi'}R - \hat{c}_1 > \overline{W}_0^c + R - \hat{c}_1 \quad \text{(A3.12)}$$

So, I get $\varphi'(\overline{W}_1^c - \overline{W}_0^c) > R$ and $(1-\varphi')(\overline{W}_1^c - \overline{W}_0^c) > P$ (A3.13). Replacing R and P with $R^c(\eta)$ and $P^c(\eta)$, and considering $R^c(0) = \varphi'(\overline{W}_1^c - \overline{W}_0^c)$ and $P^c(0) = (1-\varphi')(\overline{W}_1^c - \overline{W}_0^c)$, (A3.12) is equivalent with $R^{c}{}'(0) < 0, P^{c}{}'(0) < 0$. The proof is now done.

Proof of (ii): Quite similar to proof of (i). Maximizing (5.6) subject to (5.5), in stead of subject to (5.4), with regard to R (and implicitly P) and c_1, I have the first order condition, with regard to R, for some value of c_1 such that $\hat{c}_1 = \arg\max_{c_1,P,R} \tilde{\tilde{v}}^c(P,R,\varphi',\beta,\eta,r,\overline{W}^c)$ s.t. (5.5):

$$1 = \frac{u'\left((1+r)(\overline{W}_1^c - \frac{(1-\varphi')(1-\eta)}{\varphi'}R - \hat{c}_1)\right)}{u'\left((1+r)(\overline{W}_0^c + R - \hat{c}_1)\right)} \quad \text{(A3.13)}$$

Then I have:

$$\frac{\varphi'}{(1-\varphi')(1-\eta)+\varphi'}(\overline{W}_1^{\,c}-\overline{W}_0^{\,c}) = R^c(\eta) \quad \text{and} \quad \frac{(1-\varphi')(1-\eta)}{(1-\varphi')(1-\eta)+\varphi'}(\overline{W}_1^{\,c}-\overline{W}_0^{\,c}) = P^c(\eta)$$

(A3.14). Considering $R^c(0) = \varphi'(\overline{W}_1^{\,c}-\overline{W}_0^{\,c})$ and $P^c(0) = (1-\varphi')(\overline{W}_1^{\,c}-\overline{W}_0^{\,c})$, (A3.14) implies $R^c{}'(0) > 0, P^c{}'(0) < 0$. The proof is now done.

<u>Proof of Proposition 6</u>:

Proof of (i): Parents' associated indirect utility is given by (2.3a'). Maximizing (2.3a') subject to (5.8) with regard to R (and implicitly P), I have the first order condition:

$$\begin{aligned}&-u'\left(A^p - R\varphi/((1-\varphi)(1-\zeta))\right)\\&+\beta(1+r)[\varphi+(1-\varphi)(1-\zeta)]u'\left((1+r)R[\varphi+(1-\varphi)(1-\zeta)]/((1-\varphi)(1-\zeta))\right)=0\end{aligned} \qquad \text{(A3.15)}$$

Replacing R with $R(\zeta)$ as a function of ζ in (A3.15), differentiating the equality with regard to ζ, and implementing the comparative statics immediately produce the following equation:

$$\begin{aligned}&R'(\zeta)\{(\varphi/((1-\varphi)(1-\zeta)))u''\left(A^p - R(\zeta)\varphi/((1-\varphi)(1-\zeta))\right)\\&+\beta(1+r)^2([\varphi+(1-\varphi)(1-\zeta)]^2/((1-\varphi)(1-\zeta)))u''\left((1+r)R(\zeta)[\varphi+(1-\varphi)(1-\zeta)]/((1-\varphi)(1-\zeta))\right)\}\\&=-(\varphi/((1-\varphi)(1-\zeta)^2))R(\zeta)u''\left(A^p - R(\zeta)\varphi/((1-\varphi)(1-\zeta))\right)\\&+\beta(1+r)(1-\varphi)u'\left((1+r)R(\zeta)[\varphi+(1-\varphi)(1-\zeta)]/((1-\varphi)(1-\zeta))\right)\\&-\beta(1+r)^2([\varphi+(1-\varphi)(1-\zeta)]^2\varphi/((1-\varphi)(1-\zeta)^2))R(\zeta)u''\left((1+r)R(\zeta)[\varphi+(1-\varphi)(1-\zeta)]/((1-\varphi)(1-\zeta))\right)\end{aligned}$$

(A3.16)

Considering $u' > 0$, $u'' < 0$, clearly, R.H.S is positive, and the coefficient of $R'(\eta)$ in L.H.S is negative. So, replacing $R(\zeta)$ with $\widetilde{R}^p(\zeta)$, the proof is done.

Proof of (ii): Parents' associated indirect utility is given by (2.3a'). Maximizing (2.3a') subject to (5.8) with regard to P (and implicitly R), I have the first order condition:

$$\begin{aligned} & -u'\left(A^p - P\right) \\ & + \beta(1+r)[\varphi + (1-\varphi)(1-\zeta)]u'\left((1+r)P[\varphi + (1-\varphi)(1-\zeta)]/\varphi\right) = 0 \end{aligned} \tag{A3.17}$$

Replacing P with $P(\zeta)$ as a function of ζ in (A3.17), differentiating the equality with regard to ζ, and implementing the comparative statics immediately produce the following equation:

$$\begin{aligned} & P'(\zeta)\{u''\left(A^p - P(\zeta)\right) \\ & + \beta(1+r)^2[\varphi + (1-\varphi)(1-\zeta)]^2 u''\left((1+r)P(\zeta)[\varphi + (1-\varphi)(1-\zeta)]/\varphi\right)\} \\ & = \beta(1+r)(1-\varphi)u'\left((1+r)P(\zeta)[\varphi + (1-\varphi)(1-\zeta)]/\varphi\right) \\ & + \beta(1+r)^2[\varphi + (1-\varphi)(1-\zeta)]((1-\varphi)/\varphi)P(\zeta)u''\left((1+r)P(\zeta)[\varphi + (1-\varphi)(1-\zeta)]/\varphi\right) \end{aligned}$$

(A3.18)

Considering $u''<0$, clearly, the coefficient of $P'(\eta)$ in L.H.S is negative. On the other hand, R.H.S can be rewritten as:

$$R.H.S = \beta(1+r)(1-\varphi)\left(u'(\mathrm{X}) + \mathrm{X}u''(\mathrm{X})\right) \tag{A3.19}$$

where $\mathrm{X} \equiv (1+r)([\varphi + (1-\varphi)(1-\zeta)]/\varphi)P(\zeta)$

Considering the form of utility (2.1), the sign of R.H.S is positive, zero and negative, corresponding to $\sigma < 1$, $\sigma = 1$ and $\sigma > 1$, respectively. So, replacing $P(\zeta)$ with $\widetilde{P}^p(\zeta)$, the proof is done.

Proof of Proposition 7:

Proof of (i): Children's associated indirect utility including default risk is given in (2.3b').

Maximizing (2.3b') subject to (5.9) with regard to R (and implicitly P) and c_1, I have the first order condition, with regard to R, for some value of c_1 such that $\hat{c}_1 = \underset{c_1,P,R}{\arg\max}\, \widetilde{v}^c(P,R,\varphi',\beta,r,\overline{W}^c)$ s.t. (5.9):

$$1-\zeta = \frac{u'\left((1+r)(\overline{W}_1^c - \frac{(1-\varphi')}{\varphi'(1-\zeta)}R - \hat{c}_1)\right)}{u'\left((1+r)(\overline{W}_0^c + R - \hat{c}_1)\right)} \qquad \text{(A3.20)}$$

Since $u'()$ is a decreasing function ($u''<0$), for $\eta > 0$ I have:

$$\overline{W}_1^c - \frac{(1-\varphi')}{\varphi'(1-\zeta)}R - \hat{c}_1 > \overline{W}_0^c + R - \hat{c}_1 \qquad \text{(A3.21)}$$

So, I get $\frac{\varphi'(1-\zeta)}{(1-\varphi')+\varphi'(1-\zeta)}(\overline{W}_1^c - \overline{W}_0^c) > R$ (A3.22). L.H.S of (A3.22) is decreasing with regard to ζ. Replacing R with $\widetilde{R}^c(\zeta)$, and considering $\widetilde{R}^c(0) = \varphi'(\overline{W}_1^c - \overline{W}_0^c)$, (A3.22) is equivalent with $\widetilde{R}^{c}{}'(0) < 0$. The proof is now done. Unlike Proposition 5, the sign of $\widetilde{P}^{c}{}'(0)$ is still uncertain.

"On the Implications of Two-sided Altruism in Human Capital Based OLG Model"

Abstract

This article summarizes some propositions regarding economic dynamics and implications of two-sided altruism, on the basis of the human-capital-based OLG model of Ehrlich and Lui (1991), with application of a modified, fertility-endogenized definition of linearly separable two-sided altruism by Abel (1987) and Altig and Davis (1993). Some properties in both transition processes and steady states, the effects of unfunded social security on an equilibrium path, and the implications in the context of so called the Samaritan's dilemma and the Recardian equivalence are also analyzed. Finally some policy implications are summarized. Our calibration results and analyses show that (1) the combination of altruism toward parents and children is crucial for determining a threshold level of initial human capital and education technology in a transition process (stagnant to growth or growth to stagnant) with an irreversible hysteresis aspect, (2) in this human-capital-based OLG model, a regular recursive induction approach might still cause inefficiency in terms of an ex-post Pareto optimality criterion as of two periods later, even if strategic effects for after children (two generations later) are appropriately taken account of, and (3) unfunded social security, which involves actuarially fair insurance as well as certainty premium transfer, does *not* affect either saving or critical values for regime change so drastically, in this two-sided altruistic economy.

1. Introduction

The objective of this article is to extract some economic implications of "two-sided altruism" in the context of intergenerational linkage on a human-capital-based, overlapping generations growth model, majorly from the viewpoints of regime change (growth and stagnancy), allocation efficiency and the effects of unfunded social security (PAYG) system. To see the reasons why two-sided altruism assumption is crucial in a typical OLG model, assume the OLG model consisting of three life stages, C, Y, O (childhood, young adulthood (middle age) and old adulthood (retirement stage)). One generation, in order to determine its own life strategies, including the number of children and human capital investment per child in a recursive induction at the beginning of young adulthood (period t) , needs to specify the way for linkage with at least one of its adjacent generations, forwardly, which determines the amount of intergenerational transfer. However, in general, this linkage (λ as defined later) *cannot* necessarily be left for a spontaneous bargaining mechanism between these two generations, simply because a positive utility gain from a positive transfer for one generation implies a negative utility gain for another generation.

Therefore we assume that fixed *altruistic* weights (proportions) of utilities between neighbor generations are exogenously given, although they might be variable in the long run. Consider the following forward altruistic utility of generation (vintage) t:

$$\begin{aligned} V_t &= u_t + \lambda u_{t+1} \\ &= \left\{u_t^{(y)} + \delta u_{t+1}^{(o)}\right\} + \lambda\left\{u_{t+1}^{(y)} + \delta u_{t+2}^{(o)}\right\} \end{aligned} \tag{1.1}$$

Here $u_t \equiv \left\{u_t^{(y)} + \delta u_{t+1}^{(o)}\right\}$, $u_t^{(y)}$ and $u_{t+1}^{(o)}$ are the whole life, young and old adulthood utility of generation t, respectively. δ is a time preference discount factor for old (retirement) stage, and λ is a *weight of altruism* toward children (next generation $t+1$). Then generation t decides its life strategies at the beginning of period t, so that they might maximize its (altruistic) utility, while consequently the next generation $t+1$ faces an implicit restriction parental generation imposed regarding the ratio in marginal utility of intergenerational transfer (bequest/compensation) between $u_{t+1}^{(o)}$ and $u_{t+1}^{(y)}$. In other words, the generation holds, in general, *two chances* of intergenerational linkage, firstly through fertility and human capital investment decision planned by middle age parent during young adulthood, and secondly through transfer (compensation/bequest) during old stage. Therefore, next generation $t+1$ is assumed, at the beginning of Y (period $t+1$), to take this implicit restriction as given, or to internalize it by adding an old adulthood utility part

$u_{t+1}^{(o)}$ of parental generation t weighted by some altruistic coefficient, λ'.

However, this second-time linkage between old-age parents and young adulthood children needs much consideration from various viewpoints. First, at this stage both adjacent generations are considered, quite naturally, to hold *independent minds*, whether altruistic or egoistic with each other in terms of utility. Therefore there might be better invented some judgmental reconciliation between these generations. Thus, as one possibility, the intergenerational bargaining during this stage might be equally "competitive", if multiple goods or actions are traded, so that the value of altruistic coefficient, λ', might no longer make any sense. Also it could take the mixed form of both altruism and competition. Second point is the existence of mortality risk in old age and income risk in young adulthood. For example, setting Arrow-Debreu prices (contingent claims) on these risks enables the two generations to behave competitively and to share the risks, even with only one traded goods (e.g. monetary transfer) or one action. Here, however, still occurs one discussion.

Under intergenerational risk sharing mechanism, bequest (from parents to children) and compensation (vice versa) correspond with each revelation of mortality/income risks. Mortality risk of parental generation generates a peculiar shape of its indifference curve, and this shape makes it for both parents and children impossible to set initially some value for the state contingent claim between two states of mortality risk, or equivalently to set the initial relative price between bequest and compensation. This is a totally different point from Arrow-Debreu state-contingent exchange economy, in which state contingent claim (or state price) enables them to arrive at a market-clearing and Pareto optimal equilibrium. As a consequence, in this short-run bargaining framework, an automatic price adjustment process to a unique equilibrium point cannot necessarily be expected, as far as any additional restrictions (e.g., regarding the marginal rate of substitution between bequest and compensation, or the proportion in the marginal utility of transfer) are not introduced. On the other hand, if a fixed level of (almost actuarially fair) social/market insurance is available for old parents, setting state prices competitively is possible. Therefore it still remains to be a question whether parental old-age transfer is effectively implemented on altruism or competitive bargaining, while this article basically focuses on the role of altruism.[16]

Now two-sided altruistic utility of the following form makes sense, so that it enables the recursive computation of value function without intergenerational bargaining confliction:

$$V_t = \lambda' u_t^{(o)} + u_t + \lambda u_{t+1} = \lambda' u_t^{(o)} + \{u_t^{(y)} + \delta u_{t+1}^{(o)}\} + \lambda u_{t+1} \qquad (1.2)$$

Here the combination of forward/backward altruism, (λ, λ'), is a kind of social norm, which supposedly holds somehow an economic sense as examined by, for example, Kandori (1992). Since, as a natural result, this form, ranging over at least two (actually three in this case) adjacent periods, contains a parental utility part $u_t^{(o)}$, that of next generation $u_{t+1} = u_{t+1}^{(y)} + \delta u_{t+2}^{(o)}$ as well as its own utility part u_t, the generation maximizes (1.2), taking account of subsequent effects its own life strategies brings on the next generation. While this (or a similar) form of two-sided altruism is examined by Abel (1985), Kimball (1987), Altig and Steve (1991), Hori (1997), Wigniolle (2002), Blackburn and Cipriani (2005), Boldrin, De Nardi and Jones (2005) and Raut (2006), our article tries to pay more consideration to the role of altruism toward parents (λ') as a rear rudder for attaining growth and/or efficiency, as well as to the timing of determination of life strategies, and a rebound effect on its old age

[16] Another possibility is Nash equilibrium as assumed in some related papers (e.g., Lindbeck et al. (1988), partially Coate (1995)). However, introducing (subgame perfect) Nash equilibrium within the model does not inherently exclude the possibility to lead to Pareto inefficiency even in symmetric (simultaneous) bargaining. (e.g., Prisoner's dilemma)

utility. If the generation is *dynamically consistent* in the sense that it behaves as parents expected in terms of transfer motive during old (retirement) stage, it must hold that $\lambda' = \delta / \lambda$, which implies that the generation, who holds less altruism toward children, should be more altruistic toward parents. However, in general, there is no guarantee that each generation holds a dynamically consistent altruism toward its parental generation, or that above all dynamic consistency leads to dynamic efficiency, especially where the economy is not in a steady state. On the other hand, from socio-biological point of view, it is rather natural to assume $\lambda' = \lambda\delta$ instead, because an *egoistic* person is normally egoistic both for parents and children, therefore tends to hold smaller λ' as smaller λ is. Thus the dynamic consistency is a concept somewhat contrary to our intuition in socio-biological context.[17] Therefore, in this article, we are going to summarize some propositions and features regarding economic dynamics and implications of two-sided altruism, being careful enough for these crucial aspects, specification of intergenerational linkage, timing of strategy determination and indirect strategic effects, effects of dynamic consistency/inconsistency and steady/unsteady states, and in addition, roles of mandatory intergenerational transfer between old parents and young adulthood as implemented in the form of unfunded social security. This is exactly the objective of this article.

The inter/intra-generational strategic interaction based on altruism has been ever discussed in a broad range of literature, for example, Becker (1976, 1981, 1988), Hirshleifer (1985), Bernheim, Shleifer and Summers (1985), Bruce and Waldman (1990), Coate (1995), Lindbeck and Weibull (1988), in the context of so called the rotten kid theorem or the Samaritan's dilemma. Also Bernheim et al. (1985), Abel (1987), Hurd (1989), Altig et al. (1993), Bernheim (1991) analyze the bequest motives. While considering two stage

[17] According to Abel (1987), Bernheim had already argued that there is no reason to insist on dynamic consistency in modeling the consumption and transfer behavior of families.

overlapping generations model as in Ehrlich and Lui (1991) and many others, our article tries to regard the combination of forward/backward altruism as a kind of sustainable social norm, and to encompass all the intergenerational interaction within one generation's decision making problem without any bargaining confliction with neighbor generations. Likewise our settings offer a convenient tool to analyze the relationship between growth and Pareto efficiency, as in Bernheim and Ray (1987), and the attributes of critical values between two regimes.

A simple, but essential question arises still on debate. "Does the form of altruistic utility really, and how if so, affect allocation efficiency?" Among many related results, one important result of the rotten kid theorem by Becker (1981) is that "each child (beneficiary), no matter how selfish, would maximize the family income of his benefactor, and thereby would internalize all effects of his actions on other beneficiaries". An interpreted implication is: *if both parents and children negotiate in an equally competitive manner, the form of altruistic utility, whether one sided or two sided or entirely egoistic, does not shift the equilibrium allocation, as far as the weight of altruism does not turn over the indifference curve of altruist to the other side and the transfer mechanism is fully operative.* On the other hand, Bernheim et al. (1985), Lindbeck et al. (1988) , Bruce et al. (1990), Coate (1995) consider various sequential decision flows with strategic interaction under altruistic parents (or the rich) and egoistic children (or the poor), and derive some inefficiency implications as in the Samaritan's dilemma. However, while each of them assumes each distinct strategic structure as exogenously given, one important argument with respect to why such asymmetric decision flows have economic rationale is still ambiguous. In addition, these diversified results derived in the above literature suggest that the implications of altruism are very sensitive to the following critical points. (1) Structure of decision flow (simultaneous or sequential? how ordered?), (2) form of utility (altruistic or egoistic?, two sided or one sided?), (3) operability of decision in the form of actions or monetary transfers, and (4) attribute of bargaining (e.g., competition in a Arrow-Debreu sense, mutual agreement on altruism and participation constraint, or (subgame perfect) Nash equilibrium?). Using a modified model of two sided altruism, this article also tries to give a partial answer to this issue, by means of discussing sustainability, enforceability and optimality of assumed social norm from both static and dynamic context.

Another important factor is, again, the existence of mortality risk, under which Abel (1985) and Hurd (1989) analyze accidental bequest motives. Furthermore, Kotlikoff and Spivak (1981) discuss the role of the family as an incomplete annuity insurance, and Kotlikoff, Shoven and Spivak (1986) does the effects of annuity insurance on savings and inequality. The remaining crucial issue is the effect of taxation and governmental bond financing social security, as in Barro's (1974) Recardian neutrality result, its rebuttal by Feldstein (1976) and Buchannan (1976), or Drazen's (1978) human capital argument under imperfect market. Although social security has miscellaneous definitions, we focused on the role of actuarially fair insurance rather than intergenerational mandatory transfer, and investigated its effects on life strategies. Furthermore, recent tendencies toward less fertility and (presumably) increasing inequality in developed countries have been discussed in the possibly causal relation of unfunded social security, computationally by Fuster (1999) or Ehrlich and Lui (1998), and empirically by Boldrin et al. (2005) or Ehrlich and Kim (2007). One neglected point in the existing literature is to be explicitly conscious of the possible other external effects exposed from parental generation, for example, of relative population and saving to revenue ratio exposed, and to clarify these causalities, together with that of social security, in more rigorous terms. Our two sided altruistic model offers more appropriate foundation for enabling it, by incorporating these (probably) causal factors into state variables. Thus this article is just comprehensively revisiting these issues as described above, and is raising some questions and propositions.

The organization of this article is as follows. Section 2 formulates the generation's problem under two-sided altruistic utility, and describes some typical characteristics, mainly based on calibration results. We analyze the model in the interrelation of static Pareto efficiency criteria in section 3, and of dynamic development in section 4. In section 5 the effects of unfunded social security tax on the economy are discussed especially with respect to regime changes and the effects on saving and education. Section 6 summarizes some crucial discussions, and shows at last some regression results which partially support our two-sided altruistic model.

2. The model and some calibration results

The model assumes, as constructed in Ehrlich and Lui (1991) and others, three stages in one generation's whole life, childhood, young and old adulthood, and adjacent generations (parents and children) sharing the same period t within adjacent life stages respectively. Young adult determines, at the beginning, some life strategies, fertility (the number of children per parent) n_t, human capital investment for each child h_t, compensation rate ω_t to be given by their children as old age support, and saving rate s_t. Here we examine the simplest case in which there exists only one representative (identical) family, and there do not exist any inter/intra-generational human-capital externalities from outside families.

We adopt the following *additively separable* form of two-sided altruism, which incorporates fertility decisions, altruistic elasticity and covers all the relevant utilities of two adjacent generations during two adjacent periods:

$$
\begin{aligned}
V_t &= \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \hat{\lambda}(n_t)u_{t+1}^{(y)} \\
&= \hat{\lambda}'(n_{t-1})u(c_{2,t}) + \{u(c_{1,t}) + \delta\pi_2 u(c_{2,t+1})\} + \hat{\lambda}(n_t)u(c_{1,t+1})
\end{aligned}
\tag{2.1}
$$

$c_{1,t}$: Young adulthood consumption at period t. $c_{2,t+1}$: Old adulthood consumption at period $t+1$. u_t: Whole life utility (of both young and old adulthood) of generation t. $u_t^{(y)}$: Young-adulthood utility of the generation at period t. $u_t^{(o)}$: Old-adulthood utility of parental generation $t-1$. $u_t^{(y)} \equiv u(c_{1,t})$, $u_{t+1}^{(o)} \equiv u(c_{2,t+1})$, $u_t \equiv u_t^{(y)} + \delta\pi_2 u_{t+1}^{(o)}$, $\hat{\lambda}(n_t) \equiv \lambda\pi_1 a(n_t)n_t$, $\hat{\lambda}'(n_{t-1}) \equiv \dfrac{\lambda'\delta\pi_2}{\pi_1 a(n_{t-1})n_{t-1}}$, and $a(n) \equiv n^{-\varepsilon}$ $(0 \le \varepsilon \le 1)$.[18] δ: Time preference discount factor for old (retirement) stage. π_1, π_2: Survival rate of young and old adulthood, respectively.

$\lambda(>0)$, $\lambda'(>0)$ and ε respectively denote the degree of pure altruism toward children (next generation $t+1$), toward parents (previous generation $t-1$), and the constant elasticity of altruism per child as their number increases. *Dynamic consistency* requires the condition $\lambda\lambda' = 1$, while *socio-biological* consistency does $\lambda = \lambda'$, but we do not impose these conditions as a general restriction. Also, define an "altruistic" economy as $\lambda\lambda' > 1$ and an "egoistic" one as $\lambda\lambda' < 1$. Unlike ordinary time preference, we do not exclude the possibility of $\lambda \ge 1$, because parents are, sometimes as an important case, very likely to put more weight in children than in themselves. Throughout this article, we consider the following three cases. *Case-A: Saving but no compensation economy, Case-B: Compensation but no saving economy,* and *Case-C: Both saving and compensation economy.*

In case A, there is no intergenerational linkage between old parents and young adults, in which these two neighbor generations are considered to be almost competitive. Figure 1 shows the specification of intergenerational linkage in case C. In this case, generation t solves the following problem:

[18] The elasticity of altruism per child (ε) was introduced by Becker et al. (1990). $\varepsilon = 1.0$ denotes perfect inelasticity of altruism with number of child.

$$\widetilde{V}_t(H_t, n_{t-1}, S_{t-1}) = \max_{\{h_t, n_t, \omega_t, s_t\}} V_t$$

$$= \max_{\{h_t, n_t, \omega_t, s_t\}} \left(\frac{(\delta\lambda')\pi_2}{(n_{t-1})^{1-\varepsilon}} u(c_{2,t}) + [u(c_{1,t}) + \delta\pi_2 u(c_{2,t+1})] + \lambda(n_t)^{1-\varepsilon} \pi_1 u(c_{1,t+1}) \right) \quad (2.2)$$

s.t. $h_{t+1} = h_{t+1}(H_{t+1}, n_t, S_t)$, $n_{t+1} = n_{t+1}(H_{t+1}, n_t, S_t)$,

$\omega_{t+1} = \omega_{t+1}(H_{t+1}, n_t, S_t)$, $s_{t+1} = s_{t+1}(H_{t+1}, n_t, S_t)$

Here state variables are H_t, n_{t-1}, S_{t-1} and control variables are n_t, h_t, ω_t, s_t. First, in this case, the determination of control variables crucially depends on altruism toward parents λ', as well as altruism toward children λ, time preference δ, interest rate D, and other parameters like σ, ε, v, $\overline{H}$, and state variables. Now we apply, to the above dynamics, the following model as in Becker et al. (1990), and Ehrlich et al. (1991).

$$y_t = [(1 - vn_t - h_t n_t)(H_t + \overline{H})]^\gamma \qquad \gamma = 1 \quad (2.3\text{-a})$$

$$H_{t+1} = A(H_t + \overline{H})h_t \ (2.3\text{-b}), \quad S_t = D\eta_t[H_t + \overline{H}](s_t)^m \ (2.3\text{-c}) \ \text{(Law of motion)}$$

$$c_{1,t} = (1 - vn_t - h_t n_t - s_t)\eta_t(H_t + \overline{H}) - \omega_t \pi_2 \eta_t(H_t + \overline{H}) \quad (2.3\text{-d})$$

$$c_{2,t+1} = \omega_{t+1}\pi_1 n_t[\eta_{t+1}(H_{t+1} + \overline{H})] + D\eta_t(H_t + \overline{H})(s_t)^m \quad (2.3\text{-e})$$

$$u_t^{(y)} \equiv u(c_{1,t}) = \left(\frac{1}{1-\sigma}\right) c_{1,t}^{\ 1-\sigma}, \quad u_{t+1}^{(o)} \equiv u(c_{2,t+1}) = \left(\frac{1}{1-\sigma}\right) c_{2,t+1}^{\ 1-\sigma} \quad (2.3\text{-f})$$

$$N_{t+1} = N_t n_t \quad (2.3\text{-g})$$

with constraints: $n_t \geq 0$, $h_t \geq 0$, $s_t \geq 0$, $y_t \geq 0$ and $c_{1,t} \geq 0$, $c_{2,t+1} \geq 0$.

$u(c) \equiv (1/(1-\sigma))c^{1-\sigma}$: Utility where σ represents the inverse value of the inter-temporal elasticity of substitution or the constant relative risk aversion coefficient. t: Period t. y, o, c: young/old adulthood, children. N_t: Number of couple (young adult) at period t. n_t: Number of children (fertility) the generation t bears per parent. y_t: Production by generation t. H_t: Acquired human capital. $\overline{H}$: Raw labor capital. (Therefore, $H_t + \overline{H}$ represents total human capital.) S_{t-1}: Total saving. h_t: Time allocation a parent (young adult) devotes to educate each child. v: Allocation of labor time a parent (young adult) devotes to raise each child. s_t: Saving rate, a fraction of income $\eta_t(H_t + \overline{H})$ allocated to saving during young adulthood. ω_t : Compensation rate from a young adult (a child) to an old adult (a parent). η_t : Wage for a unit of human capital.[19] A : Education efficiency rate (education technology). D : Rate of return in saving.

Now we show computation results for some parameter values, using programs for recursive dynamic computation. Set the values at $\gamma = 1.0$, $\sigma = 0.5$, $\varepsilon = 0.5$, $\delta = 0.5$, $D = 2.0$, $\nu = 0.1$, $\overline{H} = 1.0$, $\varsigma = \rho = 1.0$ and $\pi_1 = \pi_2 = 1.0$ (perfect foresight). Also, for the convenience of calibration, we limit the values of life strategies chosen, as following:

$h_{\min} = 0 \le h_t \le 1.0 = h_{\max}$, $s_{\min} = 0 \le s_t \le 0.5 = s_{\max}$, $n_{\min} = 0.5 \le n_t \le 2 = n_{\max}$ [20]

and $\omega_{\min} = -0.5 \le \omega_t \le 0.5 = \omega_{\max}$ (only in case B and C) (2.4)

Hereafter, we use these parameter settings summarized in Table 1 as default values, as far as there are no special comments. This assumption is set in order to make the calibration possible and the economy fit well in reality. In addition, we set the terminal condition, $h_{T+1} = h_{\min}$, $n_{T+1} = n_{\min}$, $s_{T+1} = s_{\min}$ (and $\omega_{T+1} = \omega_{\min}$).

Here we have $\partial V_t / \partial \omega_{t+1} > 0$, < 0, $= 0$, if $\lambda\lambda' < 1$, > 1, $= 1$, respectively. Likewise, $\partial V_t / \partial n_{t+1} < 0$, $\partial V_t / \partial s_{t+1} < 0$ and $\partial V_t / \partial H_{t+2} < 0$. h_t, n_t, ω_t and s_t are not affected by the value of H_t, but are the functions only of n_{t-1} and H_t / S_{t-1}, if $H_t >> \overline{H}$, probably in a growth state, while H_t matters, if $H_t \cong \overline{H}$ or $H_t \le \overline{H}$, probably in a stagnant state. In the latter case, the relative values of H_t to $\overline{H}(=1)$ and S_{t-1}, as well as relative population n_{t-1}, are crucial determinants, in the forthcoming transition process, to overpass a stumbling block to a growth state, or to go down into a Malthusian trap. In order to enable the analysis with a clear contrast, we consider the

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[19] With linear production function ($\gamma = 1$), $\eta_t = 1.0$, assuming that human capital market is fully competitive.

[20] In reality n_t takes some discrete values, 0, 0.5, 1.0, 1.5 and 2.0. $n_t = 1$ implies that the generation bears 2 children per couple.

following four combinations of coefficients of altruism toward children and parents, (λ, λ'). I: $(\lambda, \lambda') = (0.5, 2.0)$, II: $(\lambda, \lambda') = (0.5, 0.5)$ ($\lambda\lambda' = 0.25 < 1$, III: $(\lambda, \lambda') = (2.0, 0.5)$, and IV: $(\lambda, \lambda') = (2.0, 2.0)$. We also search out the following critical values, important indices of transition process. $H^{\uparrow}_{1\,A=5}$: Critical value of initial human capital H_1, which makes the economy pushed up into a growth state, when education efficiency is at $A = 5$ ($A^{\downarrow}_{H_1=10} < A < A^{\uparrow}_{H_1=0}$).[21] $A^{\uparrow}_{H_1=0}$: Critical value of education efficiency A, which makes the economy pushed up into a growth state, when the initial human capital is the lowest at $H_1 = 0 << \overline{H} = 1$. $A^{\downarrow}_{H_1=10}$: Critical value of A, which makes the economy trapped into a stagnant state, when the initial human capital is relatively high at $H_1 = 10 >> \overline{H} = 1$.

In case A, the value function can be divided into two additively separable parts:

$$\widetilde{V}_t(H_t, n_{t-1}, S_{t-1}) = \breve{V}_t(H_t) + f(n_{t-1}, S_{t-1}) \tag{2.5}$$

As a consequence, the life strategies (h_t, n_t, s_t) of generation t depend at most only on its own human capital H_t, not on altruism toward parents λ' or parental fertility or saving decision (n_{t-1} or S_{t-1}), so that $h_t = h_t(H_t)$, $n_t = n_t(H_t)$, and $s_t = s_t(H_t)$.

Next consider case B, where there is no saving habit ($D = 0$). We set the default parameter values as in Table 1, and in addition, $n_0 = 1$. We plotted the transition paths of life strategies for case I, II, III and IV, for the same combinations of A and H_1, (a) $A = 5$ and H_1 is moving at 1.0, 3.0 and 5.0, (b) $H_1 = 0$ and A is moving at 2.5, 5.0, 7.5 and 10.0, and (c) $H_1 = 10$ and A is moving at 1.0, 2.0, 3.0 and 4.0. In each of I, II, III and IV, we observe, as in case A, that $A^{\downarrow}_{H_1=10} < A^{\uparrow}_{H_1=0}$, a *hysteresis* aspect (i.e., *irreversible threshold*

[21] $H^{\uparrow}_{1\,A=5} = H^{\downarrow}_{1\,A=5}$. $A = 5$ is taken here, because this value is situated in an intermediate region between unconditional growth regime and unconditional stagnant regime.

for regime change) of transition process, and the existence of $H^{\uparrow}_{1\,A=5}(>0)$ for $\lambda = 0.5 < 1$, one feature of "Malthusian trap". Wigniolle (2002) points out the "hysteresis" property in steady states. Also we observe a phenomenon of dynamic fluctuation in III and IV for $\lambda = 2 > 1$.[22] In general, our calibration results support the existence of two equilibria (one with larger growth and low fertility where fertility is binding, and the other with low growth and large fertility where human capital investment is binding), claimed by Becker et al. (1990) and Ehrlich et al. (1991).

The effect of initial fertility level n_{t-1} depends on the values of σ and ε. If $\varepsilon > \sigma$, lower initial fertility (small n_{t-1}) induces lower compensation rate ω_t, and through a positive income effect induces probably larger investment for human capital and fertility (h_t and n_t), being an accelerating factor of growth, while larger initial fertility might be an impediment against growth.[23] And vice versa if $\varepsilon < \sigma$. Instead, if $\varepsilon = \sigma$, then any shock in n_{t-1} is neutralized (canceled off) and does not affect the life strategies of generation t (h_t, n_t and ω_t). As for the effect of initial human capital H_t, the same property holds, in which it does not affect the subsequent life strategies under a growth regime. Summarizing the above, we have the following property. If $\varepsilon = \sigma$, then $\partial\omega_t / \partial n_{t-1} = 0$, $\partial n_t / \partial n_{t-1} = 0$ and $\partial h_t / \partial n_{t-1} = 0$ ($\partial H_{t+1} / \partial n_{t-1} = 0$). If $H_t >> \overline{H}$ and $h_t > 1/A$, then $\partial\omega_t / \partial H_t = 0$, $\partial n_t / \partial H_t = 0$ and $\partial h_t / \partial H_t = 0$ ($\partial H_{t+1} / \partial H_t \cong Ah_t \neq 0$). The economy is always in a steady state if $\varepsilon = \sigma$ (Relative risk aversion coefficient is equal to elasticity of altruism per child.) and $H_t >> \overline{H}$ (a growth regime).

One important aspect of case C is that compensation ω_t might not necessarily be

[22] Nishimura and Benhabib (1989, 1993) analyze the mechanism for endogenous fertility fluctuation in a Barro-Becker model.
[23] To be more precise, fertility could decrease, where there does occur a regime change into growth, and/or the substitution between education and fertility is remarkable.

positive but negative (i.e., a positive bequest) even if λ' is positive, under the existence of high parental saving S_{t-1}. This tendency is strengthened especially when the old-age mortality risk exists and the intergenerational transfer involves a corresponding risk sharing between the two adjacent generations (old and young adulthood). Reversely speaking, the negative expected compensation does not necessarily imply the non-existence of positive altruism. Therefore it is a delicate task to estimate econometrically λ' with a potential mixture of social/market insurance under continuous time mortality risk.[24]

If $\widetilde{\lambda}'(\equiv Dm(s_{t-1})^{m-1}/[A\lambda\{1-vn_t-h_t n_t-s_t\}])$ is far different from λ', then clearly s_t deviates from in case A, and $\omega_t \neq 0$. Next, the direct linkage between parents' old age and children's middle age through setting λ' enables the internalization of the externality from unexpected change in parents' saving to some extent, and might lead to more efficiency than in case A, as far as some of these intergenerational linkage are not binding. However, this kind of internalization might still end in insufficiency.

Effect of dynamic inconsistency ($\lambda\lambda' \neq 1$)

Assume, again, case B. As a matter of fact, dynamic inconsistency is not any problem as far as generation t predicts precisely the reaction functions of next generation, $h_{t+1}=h_{t+1}(H_{t+1},n_t)$, $n_{t+1}=n_{t+1}(H_{t+1},n_t)$ and $\omega_{t+1}=\omega_{t+1}(H_{t+1},n_t)$, and the *ex-ante* efficiency is kept in the sense that the generation maximizes its own two-sided altruistic utility, given initial human capital H_t, initial fertility n_{t-1}, and the above strategic effects on next generation. However, some points are still to be noted. At first, the

[24] In this regard, Kohara and Ohtake (2006) state that the parental care supplied by Japanese middle age is not motivated entirely by altruism. This implies that, as far as home health care service for frail or ill parents is concerned, a competitive bargaining between the two generation rather than altruism toward old parents prevails, and as a consequence, the weight of altruism λ' does not matter. Even in this case, the estimated λ', substantially replaced by $\widetilde{\lambda}'$, may not be zero but a positive value.

first order condition of (2.2) in fertility is done for n_t, not for N_{t+1}. This implies that it does not take account of the effect of its fertility decision on the welfare after $t+1$. Next, see that (2.2) is equivalent with the following problem:

$$\widetilde{V}_t(H_t, n_{t-1}) = \max_{\{h_t, n_t, \omega_t\}} \begin{pmatrix} \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t^{(y)} + (1-\lambda\lambda')\delta\pi_2 u_{t+1}^{(o)} \\ + \hat{\lambda}(n_t)\left\{\widetilde{V}_{t+1}(H_{t+1}, n_t) - \delta\pi_2 u_{t+2}^{(o)} - \hat{\lambda}(n_{t+1})u_{t+2}^{(y)}\right\} \end{pmatrix} \quad (2.2a)$$

s.t. $h_{t'} = h_{t'}(H_{t'}, n_{t'-1})$, $n_{t'} = n_{t'}(H_{t'}, n_{t'-1})$, $\omega_{t'} = \omega_{t'}(H_{t'}, n_{t'-1})$, $H_{t'+1} = A[H_{t'} + \overline{H}]h_{t'}$

$(t' = t+1, t+2)$

This is a Bellman-equation of (2.2). This form involves the utility parts (parents/children) and reaction functions until period $t+2$. So the internalization of the first order conditions within this recursive computation framework might be possibly incomplete whether the economy is in a steady state (or is in a transition process) or not, and this aspect causes somewhat inefficiency. On the other hand, from (2.2a), it is seen how altruism toward parents λ' affects value function $\widetilde{V}_t(H_t, n_{t-1})$, in which more λ' contributes in the utility part $u_t^{(o)}$, but hurts the part $u_{t+1}^{(o)}$. As far as the calibration results ((a) , (b) and (c) for Case I to IV) are concerned, more altruism either toward parents (larger λ') or toward children (larger λ) increases $\widetilde{V}_t(H_t, n_{t-1})$. As for normalized value function $\widetilde{\widetilde{V}}_t (\equiv \widetilde{V}_t/(1+\lambda+\lambda'))$, larger λ increases it, but larger λ' decreases.[25] Between I and III (both $\lambda\lambda'=1$), III ($(\lambda,\lambda')=(2.0,0.5)$) is better than I ($(\lambda,\lambda')=(0.5,2.0)$) both in $\widetilde{V}_t(H_t, n_{t-1})$ and $\widetilde{\widetilde{V}}_t(H_t, n_{t-1})$.

[25] For example, Dockner and Mosburger (2007) (*Journal of Difference Equations and Applications*, Vol. 13, Nos. 2-3, Feb.-Mar. 2007, pp. 197-215) consider a "normalized value function" with a different definition.

3. Dynamics and golden rule based on normalized value function

Consider again case B (compensation but no saving economy). The first order condition of (2.2) can be compactly rewritten in the following equations.

$$\omega_t: \qquad \frac{\partial V_t}{\partial \omega_t} \equiv V_t^{\omega_t} = 0$$

$$H_{t+1}: \qquad \frac{\partial V_t}{\partial H_{t+1}} + \frac{\partial V_t}{\partial \omega_{t+1}}\frac{\partial \omega_{t+1}}{\partial H_{t+1}} + \frac{\partial V_t}{\partial n_{t+1}}\frac{\partial n_{t+1}}{\partial H_{t+1}} + \frac{\partial V_t}{\partial H_{t+2}}\frac{\partial H_{t+2}}{\partial H_{t+1}} \equiv V_t^{H_{t+1}} \le 0$$

with equality if $H_{t+1} > 0$ ($h_t > 0$).

$$n_t: \qquad \frac{\partial V_t}{\partial n_t} + \frac{\partial V_t}{\partial \omega_{t+1}}\frac{\partial \omega_{t+1}}{\partial n_t} + \frac{\partial V_t}{\partial n_{t+1}}\frac{\partial n_{t+1}}{\partial n_t} + \frac{\partial V_t}{\partial H_{t+2}}\frac{\partial H_{t+2}}{\partial n_t} \equiv V_t^{n_t} \le 0 \qquad (3.1)$$

with equality if $n_t > 0$.

Assume that the second and third equations in (3.1) hold with equality, that is, the non-negativity constraints in human capital and fertility are not binding. With a slight change in some parameter represented as B_t, the first order total differentiation of (3.1) becomes in a matrix form:

$$X\begin{pmatrix} d\omega_t \\ dh_t \\ dn_t \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial V_t^{\omega_t}}{\partial B_t} \\ \dfrac{\partial V_t^{H_{t+1}}}{\partial B_t} \\ \dfrac{\partial V_t^{n_t}}{\partial B_t} \end{pmatrix} dB_t \quad \text{or} \quad \begin{pmatrix} d\omega_t \\ dh_t \\ dn_t \end{pmatrix} = -X^{-1}\begin{pmatrix} \dfrac{\partial V_t^{\omega_t}}{\partial B_t} \\ \dfrac{\partial V_t^{H_{t+1}}}{\partial B_t} \\ \dfrac{\partial V_t^{n_t}}{\partial B_t} \end{pmatrix} dB_t$$

$$\text{where } X \equiv (X_{ij}) \equiv \begin{pmatrix} \dfrac{\partial V_t^{\omega_t}}{\partial \omega_t} & \dfrac{\partial V_t^{\omega_t}}{\partial H_{t+1}} A[H_t + \overline{H}] & \dfrac{\partial V_t^{\omega_t}}{\partial n_t} \\ \dfrac{\partial V_t^{H_{t+1}}}{\partial \omega_t} & \dfrac{\partial V_t^{H_{t+1}}}{\partial H_{t+1}} A[H_t + \overline{H}] & \dfrac{\partial V_t^{H_{t+1}}}{\partial n_t} \\ \dfrac{\partial V_t^{n_t}}{\partial \omega_t} & \dfrac{\partial V_t^{n_t}}{\partial H_{t+1}} A[H_t + \overline{H}] & \dfrac{\partial V_t^{n_t}}{\partial n_t} \end{pmatrix}$$

$$\text{and } X^{-1} = adjX / \det X \qquad (3.2)$$[26]

If, for example, $B_t = n_{t-1}$, then $\partial V_t^{H_{t+1}} / \partial n_{t-1} = 0$ and $\partial V_t^{n_t} / \partial n_{t-1} = 0$. Therefore:

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[26] adj and $\det$ denote the adjoint and determinant of the matrix, respectively.

$d\omega_t = -(C_{11} / \det X)(\partial V_t^{\omega_t} / \partial n_{t-1})dn_{t-1}$, $dh_t = -(C_{12} / \det X)(\partial V_t^{\omega_t} / \partial n_{t-1})dn_{t-1}$ and

$dn_t = -(C_{13} / \det X)(\partial V_t^{\omega_t} / \partial n_{t-1})dn_{t-1}$, where C_{ij} is the cofactor of X. (3.3)

Clearly $d\omega_t$, dh_t and dn_t are the functions of λ as well as λ' and δ. Now the directions of marginal effects of n_{t-1} on life strategies ω_t, h_t and n_t are summarized as follows:

If $\varepsilon > \sigma$, then $\operatorname{sgn}(d\omega_t / dn_{t-1}) = \operatorname{sgn}(-C_{11} / \det X)$, $\operatorname{sgn}(dh_t / dn_{t-1}) = \operatorname{sgn}(-C_{12} / \det X)$,

and $\operatorname{sgn}(dn_t / dn_{t-1}) = \operatorname{sgn}(-C_{13} / \det X)$.

And if $\varepsilon < \sigma$, the signs are opposite.

If, on the other hand, $B_t = \lambda'$, then we have $\partial V_t^{H_{t+1}} / \partial \lambda' \neq 0$ and $\partial V_t^{n_t} / \partial \lambda' \neq 0$ although λ' is also some parameter contained in old-age parental utility part $\hat{\lambda}'(n_{t-1})u(c_{2,t})$. The reason is that next generation's strategy $\{n_{t+1}, h_{t+1}, \omega_{t+1}, s_{t+1}\}$ is an *explicit* function of social norm λ' as well as λ, δ and other system parameters. Now we consider the following two extreme regimes to investigate the effects of binding constraints in fertility and human capital.

(i) Growth regime and fertility is binding ($n_t = n_{\min} = 0.5$)

$$\begin{pmatrix} d\omega_t \\ dh_t \end{pmatrix} = -\frac{1}{\Delta_{-n}} \begin{pmatrix} \dfrac{\partial V_t^{H_{t+1}}}{\partial H_{t+1}} A[H_t + \overline{H}] & -\dfrac{\partial V_t^{\omega_t}}{\partial H_{t+1}} A[H_t + \overline{H}] \\ -\dfrac{\partial V_t^{H_{t+1}}}{\partial \omega_t} & \dfrac{\partial V_t^{\omega_t}}{\partial \omega_t} \end{pmatrix} \begin{pmatrix} \dfrac{\partial V_t^{\omega_t}}{\partial B_t} \\ \dfrac{\partial V_t^{H_{t+1}}}{\partial B_t} \end{pmatrix} dB_t \quad (3.2a)$$

where $\Delta_{-n} \equiv \left\{ \dfrac{\partial V_t^{\omega_t}}{\partial \omega_t} \dfrac{\partial V_t^{H_{t+1}}}{\partial H_{t+1}} - \dfrac{\partial V_t^{\omega_t}}{\partial H_{t+1}} \dfrac{\partial V_t^{H_{t+1}}}{\partial \omega_t} \right\} A[H_t + \overline{H}]$

(ii) Stagnant regime and human capital investment is binding ($h_t = h_{\min} = 0$)

$$\begin{pmatrix} d\omega_t \\ dn_t \end{pmatrix} = -\frac{1}{\Delta_{-h}} \begin{pmatrix} \frac{\partial V_t^{n_t}}{\partial n_t} & -\frac{\partial V_t^{\omega_t}}{\partial n_t} \\ -\frac{\partial V_t^{n_t}}{\partial \omega_t} & \frac{\partial V_t^{\omega_t}}{\partial \omega_t} \end{pmatrix} \begin{pmatrix} \frac{\partial V_t^{\omega_t}}{\partial B_t} \\ \frac{\partial V_t^{n_t}}{\partial B_t} \end{pmatrix} dB_t \tag{3.2b}$$

where $\Delta_{-h} \equiv \left\{ \frac{\partial V_t^{\omega_t}}{\partial \omega_t} \frac{\partial V_t^{n_t}}{\partial n_t} - \frac{\partial V_t^{\omega_t}}{\partial n_t} \frac{\partial V_t^{n_t}}{\partial \omega_t} \right\}$

Consider (i) (growth regime). $\Delta_{-n} > 0$ and $\frac{\partial V_t^{H_{t+1}}}{\partial H_{t+1}} < 0$ almost surely. $\frac{\partial V_t^{\omega_t}}{\partial \omega_t} < 0$ and $\frac{\partial V_t^{H_{t+1}}}{\partial \omega_t} = \frac{\partial V_t^{\omega_t}}{\partial H_{t+1}} < 0$. Also $\frac{\partial V_t^{\omega_t}}{\partial \lambda'} > 0$ and:

$$\begin{aligned} \frac{\partial V_t^{H_{t+1}}}{\partial \lambda'} &= \frac{\partial^2 V_t}{\partial H_{t+1} \partial \omega_{t+1}} \frac{\partial \omega_{t+1}}{\partial \lambda'} + \frac{\partial^2 V_t}{\partial H_{t+1} \partial H_{t+2}} \frac{\partial H_{t+2}}{\partial \lambda'} \\ &+ \frac{\partial V_t}{\partial \omega_{t+1}} \frac{\partial^2 \omega_{t+1}}{\partial H_{t+1} \partial \lambda'} + \left\{ \frac{\partial^2 V_t}{\partial^2 \omega_{t+1}} \frac{\partial \omega_{t+1}}{\partial \lambda'} + \frac{\partial^2 V_t}{\partial \omega_{t+1} \partial H_{t+2}} \frac{\partial H_{t+2}}{\partial \lambda'} \right\} \frac{\partial \omega_{t+1}}{\partial H_{t+1}} \\ &+ \frac{\partial V_t}{\partial H_{t+2}} \frac{\partial^2 H_{t+2}}{\partial H_{t+1} \partial \lambda'} + \left\{ \frac{\partial^2 V_t}{\partial \omega_{t+1} \partial H_{t+2}} \frac{\partial \omega_{t+1}}{\partial \lambda'} + \frac{\partial^2 V_t}{\partial^2 H_{t+2}} \frac{\partial H_{t+2}}{\partial \lambda'} \right\} \frac{\partial H_{t+2}}{\partial H_{t+1}} \\ &\neq 0 \end{aligned}$$

(Similarly in (ii) (stagnant regime), $\frac{\partial V_t^{n_t}}{\partial \lambda'} \neq 0$.) Therefore h_t is not necessarily a monotone function of λ', and an optimal growth might be attained at non-zero λ'.

Dynamic consistency and optimal choice of λ' and λ

For the sake of optimal choice of λ' and λ, normalized value function $\tilde{\tilde{V}}_t (\equiv \tilde{V}_t / (1 + \lambda + \lambda'))$ can be considered instead of $\tilde{V}_t$, because $\tilde{V}_t$ is an increasing monotone function with regard to λ' and λ, so that we have $d\tilde{\tilde{V}}_t / d\lambda' = 0$ and $d\tilde{\tilde{V}}_t / d\lambda = 0$.

On the other hand, the conditions for attaining maximal growth are:

$$dh_t / d\lambda' = -(C_{12} \partial V_t^{\omega_t} / \partial \lambda' + C_{22} \partial V_t^{H_{t+1}} / \partial \lambda' + C_{32} \partial V_t^{n_t} / \partial \lambda') / \det X = 0$$

$$dh_t / d\lambda = -(C_{22} \partial V_t^{H_{t+1}} / \partial \lambda + C_{32} \partial V_t^{n_t} / \partial \lambda) / \det X = 0 \tag{3.3}$$

Assume, for convenience, that λ is fixed and only λ' is variable. *Then the golden rule in this two-sided altruistic economy is described as*:

$$\arg\max_{\lambda'} \tilde{\tilde{V}}_t = \arg\max_{\lambda'} n_t h_t \text{, that is,}$$

$$d\tilde{\tilde{V}}_t / d\lambda' = \{(1+\lambda+\lambda')d\tilde{V}_t / d\lambda' - \tilde{V}_t\}/(1+\lambda+\lambda')^2 = 0$$

$$\Leftrightarrow d(n_t h_t)/d\lambda' = n_t(dh_t / d\lambda') + (dn_t / d\lambda')h_t = 0 \qquad (3.4)$$

In general dynamic consistency ($\lambda\lambda' = 1$) does not ensure this condition to be satisfied.

Example

Set $t = 1$. In Table 2, we calculate steady state equilibria, given initial fertility n_0, for I, II, III and IV of case B (only compensation) in both stagnant ($H_1 = 0, A = 5$) and growth regime ($H_1 = 10, A = 5$). Here we used steady states restriction, $h_2 = h_1$, $\omega_2 = \omega_1$ and $n_2 = n_1 (= n_0)$, in stead of regular reaction functions (e.g., $h_2 = h_2(H_2, n_1)$, $n_2 = n_2(H_2, n_1)$ or $\omega_2 = \omega_2(H_2, n_1)$), because regular reaction functions do not ensure either stationary or stability conditions, neither do wipe out the possibility of dynamic fluctuations. As a matter of fact, steady states might not be the equilbria the economy would eventually arrive at. Nevertheless calibrated results intuitively support the description stated above and in section 2. As easily seen, the highest h_1 (or $n_1 h_1$), the highest value and the highest normalized value do not coincide with each other. Also the higher growth of human capital proves to be achieved under an egoistic attitude toward parents ($\lambda' = 0.5$), regardless of the value of altruism toward children (λ'). It also proves, quite naturally, that h_1 is sensitive to λ (rather than to λ'), and so is ω_1 to λ'.

4. Welfare analyses

In this section we propose one efficiency criterion which enables to-the-point welfare analysis for this two-sided altruistic overlapping generations model. As already stated, each generation, who solves problems (2.2) (for case C), behaves “efficiently” as far as it precisely predicts the reaction functions of subsequent generations, whether the life strategies are in a steady state or in a transition process. (*Ex-ante optimality of generation* t *as of period* t *in terms of two-sided altruistic utility*) However, since the objective function involves utilities of two adjacent generations over just two corresponding periods, the following welfare criterion could be considered.

Criterion 1 (case C)-Ex-post Pareto optimality between generation t and t+1 as of period t+2:

Assume that period t state variables H_t, n_{t-1}, S_{t-1} and period $t+2$ state variables $H_{t+2}, n_{t+1}, S_{t+1}$ are exogenously given, and that period $t+2$ state variables $H_{t+2}, n_{t+1}, S_{t+1}$ and life strategies $\{n_{t'}, h_{t'}, \omega_{t'}, s_{t'}\}_{t'=t}^{+\infty}$ solve the problem (2.2), given period t state variables H_t, n_{t-1}, S_{t-1}. Then is the allocation, which is determined between generation t and $t+1$, as generation t control variables $\{n_t, h_t, s_t\}$ and $t+1$ control variables $\{\omega_{t+1}\}$, Pareto-efficient for generation t 's per-capita utility (u_t) and $t+1$'s altruism-adjusted aggregate utility ($\ddot{u}_{t+1} \equiv (n_t)^{1-\varepsilon} u_{t+1}$)? [27]

This criterion is a novel one in the literature. It is static analysis on *ex-post* Pareto efficiency criterion as of period $t+2$, in terms of the individual whole life utilities of two adjacent generations t and $t+1$. Then it is possible to consider some virtual and

[27] $u_t \equiv u_t^{(c)} + \delta\pi_2 u_{t+1}^{(c)}$ is *per-capita* whole-life utility of generation t. *Altruism-adjusted aggregate* utility, $\ddot{u}_{t+1} \equiv (n_t)^{1-\varepsilon} u_{t+1}$, could be replaced with per-capita whole-life utility u_{t+1} or simple *aggregate* utility $\ddot{u}_{t+1} \equiv n_t u_{t+1}$.

symmetric bargaining between u_t and $\ddot{u}_{t+1}$ in the ex-post context as of period $t+2$, as if fertility n_t, human capital investment h_t and compensation ω_t are distinct traded commodities. Bernheim et al. (1987) consider some normative properties in an aggregate neo-classical (one-period-for-one-life) growth model with intergenerational altruism, and Hori (1996) extends the analyses to two sided altruism. Our model adopts (two-periods-for one life) overlapping generations model, in which more complicated strategic interaction with neighbor generations takes place over the two periods. (Precisely n_t, h_t and ω_t at period t, and ω_{t+1} at period $t+1$ are involved in generation t's individual utility.) Therefore the corresponding normative (welfare) criterion should be adjusted so as to meet these new circumstances.

At first consider a regular representative agent (or a dynasty) model, in which each generation covers only each one period with two typical state variables, physical capital K_t and labor L_t. (e.g., $V_t(K_t, L_t) \equiv u_t + \lambda u_{t+1} + \lambda^2 u_{t+2} + \cdots$) Also assume that we are now standing at period $t+1$. Then if envelope theorems (e.g., $dV_{t+1}/dK_{t+1} = \partial u_{t+1}/\partial K_{t+1}$ or $dV_{t+1}/dL_{t+1} = \partial u_{t+1}/\partial L_{t+1}$) hold in general, the marginal effects (total derivatives) of K_{t+1} and L_{t+1} on dynastic utilities of generation t and $t+1$ (V_t, V_{t+1}) are reduced just to partial derivatives of these two generations' individual utilities (u_t, u_{t+1}) with regard to K_{t+1} and L_{t+1}. Here generation t and $t+1$ are made indifferent to the strategic (indirect) marginal effects on generation $t+2$ state variables, K_{t+2} and L_{t+2}, which are canceled off in utility over $t+1$ individual utility (u_{t+1}) and generation $t+2$'s dynastic utility (V_{t+2}). Therefore the envelope theorems are necessary conditions for *ex-ante* Pareto efficiency. However this kind of efficiency criterion has some caveats. One is that envelope theorems, in a dynasty model, might ensure to internalize indirect effects on generation $t+2$'s strategies within generation $t+1$'s individual utility, but not necessarily to cancel it off within generation $t+2$'s dynastic utility. In other words, generation $t+1$'s decision could be still a negative or positive externality for generation $t+2$. Second point is that, in a typical overlapping generations model, in which generation's entire life covers two (or three) periods, these envelope theorems do not hold basically, and besides the first order conditions also cover multiple (more than two) periods, a potential cause for ex-post distortion in welfare.[28] Our criterion sets up an adiabatic condition that period $t+2$ state variables, as well as period t variables, are kept constant, to resolve the externality problem for after-$t+1$ generations, and then Pareto efficiency in resource allocation are examined between generation t and $t+1$ with both flanks of period t and $t+2$ state variables fixed.

Let's see the issue more rigorously. In order for criterion 1 to hold, next equations for some positive coefficient κ must *necessarily* be satisfied for equating the marginal rate of substitution between u_t and $\ddot{u}_{t+1}$ among traded commodities, $N_{t+1}(n_t)$, $H_{t+1}(h_t)$ (and ω_{t+1}):

$$\partial u_t / \partial H_{t+1} + \kappa_h \partial \ddot{u}_{t+1} / \partial H_{t+1} = 0,\ \partial u_t / \partial N_{t+1} + \kappa_n \partial \ddot{u}_{t+1} / \partial N_{t+1} = 0$$

and $\partial u_t / \partial \omega_{t+1} + \kappa_\omega \partial \ddot{u}_{t+1} / \partial \omega_{t+1} = 0$ (case B and C only)

and $\kappa = \kappa_n = \kappa_h (= \kappa_\omega)$ (4.1)

Here it is assumed that these partial derivatives are implemented, keeping period t and $t+2$ state variables ($H_t, n_{t-1}, (S_{t-1})$ and $H_{t+2}, n_{t+1}, (S_{t+1})$) constant.

Defining $\ddot{u}_{t+1}^{(y)} \equiv (n_t)^{1-\varepsilon} u_{t+1}^{(y)}$, (4.1) is equivalent with:

$$H_{t+1}: \qquad \frac{\partial u_t}{\partial H_{t+1}} + \kappa_h \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial H_{t+1}} = 0$$

[28] In a special case that the economic system is symmetric between capital K_t and labor L_t, the ex-ante/post efficiency is kept regardless of whether the envelope theorems hold or not.

$$n_t: \qquad \frac{\partial u_t}{\partial n_t} + \kappa_n \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial n_t} = 0$$

$$\omega_{t+1}: \qquad \frac{\partial u_t}{\partial \omega_{t+1}} + \kappa_\omega \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial \omega_{t+1}} = 0 \qquad \text{(case B and C only)} \qquad (4.1a)$$

If population N_t and N_{t+2} are fixed, instead of fertility n_{t-1} and n_{t+1}, then the second equation of (4.1a) is replaced with: [29]

$$n_t: \qquad \frac{\partial u_t}{\partial n_t} + \kappa_n \left\{ \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial n_t} - \left(\frac{n_{t+1}}{n_t} \right) \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial n_{t+1}} \right\} = 0 \qquad (4.1b)$$

Instead, if per-capita utility $u_{t+1}^{(y)}$ represents generation $t+1$, then the second equation of (4.1a) is replaced with:

$$n_t: \qquad \frac{\partial u_t}{\partial n_t} + \kappa_n \left\{ \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial n_t} - \frac{(1-\varepsilon)\ddot{u}_{t+1}^{(y)}}{n_t} \right\} = 0 \qquad (4.1c)$$

On the other hand, (3.1) can be re-written as:

$$H_{t+1}: \qquad \frac{\partial u_t}{\partial H_{t+1}} + \lambda \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial H_{t+1}} + \lambda \left(\frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial \omega_{t+1}} \frac{\partial \omega_{t+1}}{\partial H_{t+1}} + \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial n_{t+1}} \frac{\partial n_{t+1}}{\partial H_{t+1}} + \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial H_{t+2}} \frac{\partial H_{t+2}}{\partial H_{t+1}} \right) \le 0$$

with equality if $H_{t+1} > 0$ ($h_t > 0$).

$$n_t: \qquad \frac{\partial u_t}{\partial n_t} + \lambda \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial n_t} + \lambda \left(\frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial \omega_{t+1}} \frac{\partial \omega_{t+1}}{\partial n_t} + \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial n_{t+1}} \frac{\partial n_{t+1}}{\partial n_t} + \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial H_{t+2}} \frac{\partial H_{t+2}}{\partial n_t} \right) \le 0$$

with equality if $n_t > 0$.

$$\omega_{t+1}: \qquad \frac{\partial u_t}{\partial \omega_{t+1}} + (1/\lambda') \frac{\partial \ddot{u}_{t+1}^{(y)}}{\partial \omega_{t+1}} = 0 \quad \text{(case B and C only)} \qquad (3.1a)$$

In case A, state variables, $n_t, \widetilde{S}_t$, do not affect the life strategies $\{n_{t+1}, h_{t+1}, s_{t+1}\}$, then

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[29] From (2.3g), $n_t = N_{t+1} / N_t$, $n_{t+1} = N_{t+2} / N_{t+1}$ and:

$$\frac{\partial}{\partial N_{t+1}} |_{N_t, N_{t+2} constant} = \left(\frac{1}{N_t} \right) \frac{\partial}{\partial n_t} - \left(\frac{N_{t+2}}{N_{t+1}^{\;2}} \right) \frac{\partial}{\partial n_{t+1}} = \left(\frac{1}{N_t} \right) \left(\frac{\partial}{\partial n_t} - \frac{n_{t+1}}{n_t} \frac{\partial}{\partial n_{t+1}} \right)$$

$\partial\omega_{t+1}/\partial n_t = 0$, $\partial n_{t+1}/\partial n_t = 0$ and $\partial H_{t+2}/\partial n_t = 0$. So $\kappa_n = \lambda$. Under a growth regime, $\partial\omega_{t+1}/\partial H_{t+1} = 0$, $\partial n_{t+1}/\partial H_{t+1} = 0$ and $\partial H_{t+2}/\partial H_{t+1} \cong Ah_{t+1} \neq 0$. Then some distortionary effect in ex-post sense essentially remains, where $\kappa_n = \lambda > \kappa_h$. Under this growth regime (in case A), the ex-post optimal allocation $\{n_{opt}, h_{opt}\}$ proves to satisfy $n_{opt} < n_t$ and $h_{opt} > h_t$, from $(\partial\ddot{u}_{t+1}^{(y)}/\partial H_{t+2})(\partial H_{t+2}/\partial H_{t+1}) < 0$. This is a dilemma. If population fixed condition (4.1b) is adopted, $\kappa_n < \lambda$ and the distortion is weakened. Instead, in case of (4.1c) (that per-capita utility $u_{t+1}^{(y)}$ represents generation $t+1$), $\kappa_n > \lambda$ and the distortion is strengthened. In case B, again under growth regime, $1/\lambda' = \kappa_\omega$ and $\lambda > \kappa_h$. So setting $\lambda\lambda' > 1$ ($\kappa_\omega \to \kappa_h$) enables the marginal rates of substitution between h_t and ω_t to be closer across the two generations. Also we have $\partial\omega_{t+1}/\partial n_t >,=,< 0$, if $\varepsilon >,=,< \sigma$, respectively, and assume that $\partial n_{t+1}/\partial n_t$ and $\partial H_{t+2}/\partial n_t$ are negligibly small. Then the marginal rates of substitution between n_t and ω_t are more closely equated, if $\lambda\lambda' > 1$ for $\varepsilon > \sigma$ or $\lambda\lambda' < 1$ for $\varepsilon < \sigma$. Therefore, where transfers in all of h_t, n_t and ω_t are not binding, a potential possibility for ex-post Pareto efficiency is left for case $\lambda\lambda' > 1$ and $\varepsilon > \sigma$. In case C, general inefficiency in terms of the above criterion arises from one simple aspect that the old-age utility of parental generation, $u_t^{(o)} (= u(c_{2,t}))$, is not additively separable in terms of saving part $S_{t-1} = D[H_{t-1} + \overline{H}](s_{t-1})^m$ and compensation part $\omega_t n_{t-1}(H_t + \overline{H})$, hence other state variables, n_{t-1}, S_{t-1}, as well as H_t, *do* affect the life strategies $\{n_t, h_t, \omega_t, s_t\}$. Briefly one important implication of the above analysis is that:

Either in case A, B or C, or either in a growth or stagnant regime, dynamic consistency ($\lambda\lambda' = 1$) might not necessarily ensure the ex-post Pareto efficiency.

In spite of perfect foresight with perfect certainty assumed in these economies, the ex-post efficiency (as of period $t+2$) is not achieved in general except for special cases, while the ex-ante efficiency (as of period t) always holds. On the other hand, in case that either human capital investment or fertility is binding, it is possible to choose some λ' for ex-post sub-optimality. As a matter of course, the above virtual thought experiment, using ex-post Pareto optimality criterion, contains some essential limitations that such symmetric bargaining between generation t and $t+1$ (in infancy) is actually unrealizable, and that searching for Pareto improving parameters including $\lambda, \lambda', \varepsilon, \sigma$ itself necessarily moves the values of the period $t+2$ state variables $H_{t+2}, n_{t+1}, S_{t+1}$, which are assumed to be exogenously given.

5. Application- Effects of unfunded social security

This section examines the impact of defined benefit type unfunded social security (PAYG) taxation on equilibrium paths and regime change, enough drawing a line between intergenerational certainty premium transfer and actuarially fair insurance. As for the analysis regarding the effects of social security under forward and/or backward altruism, we have, for example, Altig et al. (1993), Ehrlich et al. (1998, 2007), Fuster (1999), Wigniolle (2002), Blackburn et al. (2005), Cigno et al. (1996), Boldrin et al. (2002, 2005). Also Nishimura et al. (1992, 1995) analyze unfunded social security with endogenized fertility. Iwamoto (2006) describes social security system in Japan, and suggests an economic role of actuarially fair insurance.

Construction of old-age pension scheme

The old-age public pension (social security) scheme can be represented by a set $(S_{t-1}^{ss}, \Omega_t^{ss})$, where $S_{t-1}(=S_{t-1}^{fa}+S_{t-1}^{ss})$ is overall saving and $\Omega_t(=\Omega_t^{fa}+\Omega_t^{ss})$ is overall compensation (intergenerational transfer from young adults to old parents). Here superscript fa and ss denote an intra-family (private), and a social security (public pension) part, respectively. Therefore S_{t-1}^{ss}, S_{t-1}^{fa}, Ω_t^{ss} and Ω_t^{fa} represent funded pension, private saving, unfunded pension, and intra-family compensation, respectively. Ω_t, Ω_t^{ss} and Ω_t^{fa} can be divided into two parts, premium (certainty) part, $\overline{\Omega}_t$, $\overline{\Omega}_t^{ss}$ and $\overline{\Omega}_t^{fa}$, and actuarially fair risk sharing (insurance) part, $\widetilde{\Omega}_t$, $\widetilde{\Omega}_t^{ss}$ and $\widetilde{\Omega}_t^{fa}$. (i.e., $\Omega_t=\overline{\Omega}_t+\widetilde{\Omega}_t$, $\Omega_t^{ss}=\overline{\Omega}_t^{ss}+\widetilde{\Omega}_t^{ss}$, $\Omega_t^{fa}=\overline{\Omega}_t^{fa}+\widetilde{\Omega}_t^{fa}$, where $E\widetilde{\Omega}_t=E\widetilde{\Omega}_t^{ss}=E\widetilde{\Omega}_t^{fa}=0$.) As is clear from the discussion so far, $(S_{t-1}^{ss}, \Omega_t^{ss})$ and $(S_{t-1}^{fa}, \Omega_t^{fa})$ do depend on the combination of altruism (λ, λ'), as well as other parameters and state variables.

Assuming that the altruism coefficients toward parents and children (λ, λ') are fixed in values, the following three cases can be categorized. (1) Two-sided altruism: $\lambda > 0$, $\lambda' > 0$ (Case B or C), (2) Forward one-way altruism: $\lambda > 0$ (Case A), and (3) Backward one-way altruism: $\lambda \cong 0$, $\lambda' > 0$ (Case B or C). Case (2) and (3) are not really interesting cases, so we omit it from analysis. As in Ehrlich and Lui (1998), we define the unfunded (PAYG) social security tax as a mandatory lump-sum transfer from the generation of young adult-hood children to old parents $T_t \equiv \theta H_t (=\overline{\Omega}_t^{ss})$, where θ is the (expected) proportional social security tax levied on middle age income part earned only by acquired human capital H_t. Then young/adult-hood consumptions of generation t are represented in the following expected terms:

$$c_{1,t}=(1-vn_t-h_t n_t-s_t)(H_t+\overline{H})-\omega_t \pi_2 (H_t+\overline{H})-T_t$$

$$c_{2,t+1} = \omega_{t+1}\pi_1 n_t (H_{t+1} + \overline{H}) + D(H_t + \overline{H})(s_t)^m + (\pi_1/\pi_2) n_t T_{t+1} \qquad (5.1)$$

Also assume $\pi_1 \cong 1$ and $\pi_2 < 1$. The *defined expected benefit,* which is transferred from children' generation to parents', amounts to $(\pi_1/\pi_2)n_t T_{t+1}$, in which fully actuarially fair insurance is also implicitly assumed. Intra-family bargaining (compensation rate ω_t) between old-age parents and young adult-hood children is motivated on altruism (λ') as a state-contingent claim only on the survival of old parents. Social security tax rate (θ) involves not only certainty premium transfer, but also actuarially fair (full or partial) insurance. Under fully actuarially fair insurance, it proves that intra-family bargaining results in only certainty premium re-transfer without risk sharing. In either way (with full or partial insurance), the existence of unfunded social security tax does not affect either overall compensation Ω_t, or subsequent equilibrium paths, as far as intra-family re-transfer is fully operative. (Altig et al. (1993))

Let's see these issues more rigorously with mortality risk sharing. (Hurd (1989), Kotlikoff et al. (1981, 1986)) Unfunded social security tax is rewritten as $T_t = \Omega_t^{ss} = \overline{\Omega}_t^{ss} + \widetilde{\Omega}_t^{ss} = \theta H_t + \widetilde{\Omega}_t^{ss}$ ($\overline{\Omega}_t^{ss} = \theta H_t$ and $E\widetilde{\Omega}_t^{ss} = 0$), and the intra-family compensation is $\Omega_t^{fa} = \overline{\Omega}_t^{fa} + \widetilde{\Omega}_t^{fa} = \omega_t(H_t + \overline{H}) + \widetilde{\Omega}_t^{fa}$. ($\overline{\Omega}_t^{fa} = \omega_t(H_t + \overline{H})$ and $E\widetilde{\Omega}_t^{fa} = 0$) In order to discuss more appropriately the effect of unfunded social security, we decompose $\Omega_t^{ss} (= \overline{\Omega}_t^{ss} + \widetilde{\Omega}_t^{ss})$ into two factors $(\overline{\Omega}_t^{ss}, \widetilde{\Omega}_t^{ss})$. This point has not necessarily been recognized fully in the literature. An interesting issue is the difference between when actuarially fair insurance and only self insurance are available, and now we focus on it.

Assume that, during the old adulthood, there exists a risk of death, which is revealed exactly at the middle point of the stage, when they are alive with probability π_2, or die with probability $1-\pi_2$. The generation holds an expected utility, which arises from *state-dependent* aspect:

$$\widetilde{u}_t^{(o)} = u_t(c_{o1}) + \delta' \pi_2 u_t(c_{o2}) \tag{5.2}$$

Here δ' is a constant time preference for each half period, and c_{o1} and c_{o2} are real consumptions of the first and second half period, respectively. The real interest rate for each half period is r. The old age parents face unfunded social security, denoted by (R_t, P_t), which finances a certainty premium transfer $k = \overline{\Omega}_t^{ss}$ from young adult generation.[30] Then *the indirect utilities* for parents regarding transfer contract schedule (R_t, P_t) are represented as:

$$\widetilde{v}^p(R_t, P_t, \pi_2, \delta', r, \widetilde{S}_{t-1}) \equiv u(\widetilde{S}_{t-1} - P_t) + \pi_2 \delta' u((1+r)(R_t + P_t)) \tag{5.3}$$

where $\delta' = \delta^{0.5}$ and $1+r = D^{0.5}$. Also set $\delta'(1+r) = 1$ for convenience. We assume two options of insurance contract for mortality risk involving this social security, *self insurance* and *actuarially fair insurance*. If social security involves self insurance, it necessarily satisfies $R_t = \overline{\Omega}_t^{ss}$, while, on the other hand, for actuarially fair insurance $\pi_2 R_t - (1-\pi_2)P_t = \overline{\Omega}_t^{ss}$. Under self insurance and actuarially fair insurance, respectively:

$$\widetilde{\widetilde{v}}_{act}^{p}(\pi_2, \delta', r, S_{t-1}, \overline{\Omega}_t^{ss}) = (1/(1-\sigma))(S_{t-1} + \overline{\Omega}_t^{ss})^{1-\sigma}(1+\pi_2\delta')^{\sigma} \quad \text{and}$$

$$\widetilde{\widetilde{v}}_{self}^{p}(\pi_2, \delta', r, S_{t-1}, \overline{\Omega}_t^{ss}) = (1/(1-\sigma))(S_{t-1} + \overline{\Omega}_t^{ss})^{1-\sigma}(1+\pi_2{}^{1/\sigma}\delta')^{\sigma} \tag{5.4}$$

For example, see Abel (1985) and Kotlikoff et al. (1986) for similar derivations. In order to compare $\sigma > 1$ and $\sigma < 1$, we need to consider another form of utility function $u(\widetilde{c}) \equiv (1/(1-\sigma))(\widetilde{c}^{1-\sigma} - 1)$ (or $u(\widetilde{c}) \equiv \ln(\widetilde{c})$ for $\sigma = 1$), where $\widetilde{c}$ is each half period

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[30] (R, P) denotes $(\mathrm{Re}\,ceipt, Payment)$. Here the left-hand side of (R, P) denotes *receipt* when old parents are alive, and the right-hand side does *payment* when they die, both sides measured by present value at the beginning of the period.

consumption, in stead of $u(c) \equiv (1/(1-\sigma))c^{1-\sigma}$, and so (5.4) does change by constant term $-(1+\pi_2\delta')/(1-\sigma)$, while the marginal utility of income itself remains unchanged. If family compensation $\overline{\Omega}_t^{fa}$ is also considered (certainty part only), then $\overline{\Omega}_t^{ss}$ can be replaced with $\overline{\Omega}_t (= \overline{\Omega}_t^{fa} + \overline{\Omega}_t^{ss})$. Here intra-family risk sharing, $\widetilde{\Omega}_t^{fa}$, is not assumed. These forms of indirect utility clarify that the marginal utility of income is the same in terms of both saving S_{t-1} and compensation $\overline{\Omega}_t^{fa}$, and in addition that it is, with actuarially fair insurance, greater than, equal to, smaller than, with self insurance, if $\sigma < 1$, $\sigma = 1$, $\sigma > 1$, respectively. Then, with $\sigma < 1$, actuarially fair social security induces, compared with self-insurance, an incentive for larger compensation $\overline{\Omega}_t^{fa}$ through effectively larger λ', and as a consequence invokes negative (disposable) income effect, while it also induces an incentive for larger next period saving S_t through effectively larger δ. In other words, with actuarially fair insurance δ is replaced with $\widetilde{\delta} = \kappa\delta$, but λ and λ' are unchanged, where $\kappa > 1$, $\kappa = 1$, $\kappa < 1$, for $\sigma < 1$, $\sigma = 1$, $\sigma > 1$, respectively. Therefore it is ambiguous whether saving rate s_t, education rate h_t and fertility n_t will be eventually larger or not, and depends on the values of λ, λ' and δ. Only case $\sigma = 1$ makes the generation be clearly indifferent between these two insurances. Therefore, in two-sided altruistic economy, it is not so straightforward to clarify analytically the direction in change of life strategies.

Example

Set $\pi_2 = 0.5$ and $t = 1$. Table 3 shows the computation results under steady states restrictions ($s_2 = s_1$, $h_2 = h_1$, $\omega_2 = \omega_1$ and $n_2 = n_1 (= n_0)$), comparing the effect of actuarially fair social security (tax rate $\theta = 0$ and variance ($Var\widetilde{\Omega}_t^{ss}$) is optimal.) and that of self-insurance case ($\theta = 0$ and $Var\widetilde{\Omega}_t^{ss} = 0$) on the economy, with $S_0 = 0.5(H_1 + \overline{H})$, two (low and high) initial human capital ($H_1 = 0{,}10$), for three cases of $\sigma = 0.5, 1.0, 2.0$, and in case C (both saving and compensation economy). Modified utility $u(\widetilde{c}) \equiv (1/(1-\sigma))(\widetilde{c}^{1-\sigma} - 1)$ is used here. Since $m = 0.5$ exhibits an increasing return to scale in saving, and so the constraints in both saving are compensation are not binding as the calibration results show. This modified utility proves to prompt less fertility even in low initial human capital ($H_1 = 0$).[31] On the other hand, we compared these two insurances with the same settings as Table 2 (Case B, no uncertainty, two regimes, steady states restrictions, $\sigma = 0.5$, regular utility $u(\widetilde{c}) \equiv (1/(1-\sigma))\widetilde{c}^{1-\sigma}$). Results exhibit the same characteristics as no uncertainty case. Together with those in Table 3, we claim the following proposition.

Assume that social security tax involves not only certainty premium transfer ($\overline{\Omega}_t^{ss} = \theta H_t$), but also incorporates actuarially fair (partially or fully) insurance of mortality risk ($\widetilde{\Omega}_t^{ss}$), furthermore that intra-family certainty premium transfer $\overline{\Omega}_t^{fa}$ is motivated on altruism (λ'). Comparing actuarially fair insurance and self insurance, there are not so drastic differences in saving s_t, fertility n_t, education h_t, therefore in the critical value for regime change. Only private compensation $\overline{\Omega}_t^{fa}$ is sensitive between these two insurances. This holds regardless of the value of relative risk aversion coefficient (σ) or elasticity of altruism (ε) or initial human capital (H_1) or the combination of altruism (λ, λ').

[31] This is clear, because in this modified utility parents would not be likely to bear the children with small consumption less than one.

This is intuitively plausible, because, under steady states restriction, the marginal changes in s_t, h_t and n_t by a slight change in $\kappa(\cong 1)$ are almost zero in the first order, irrespective of (λ, λ'). However, in a transition process (or in a dynamic fluctuation), the effects of actuarially fair insurance are still ambiguous. Another case, where parents and children are competitive with each other, can be approximated in case A (saving only). Then actuarially fair insurance increases (decreases) saving s_t for $\sigma < 1$ $(\sigma > 1)$. Education h_t and fertility n_t are more likely to decrease (increase) and increase (decrease) for $\sigma < 1$ $(\sigma > 1)$ respectively, because of the negative income effect. Yet rigorously the signs are ambiguous, because drastic substitution between education and fertility could occur in a regime change.

As for the effect of social security on the saving and inequality, a pioneering work by Kotlikoff et al. (1986) insists that perfecting annuity insurance in altruistic economies could dissipate absolute inequality. Also there is another controversy that social security decreases saving. (e.g., Feldstein (1976)) Fuster (1999) claims, mainly from calibration results based on estimated parameter values, that social security has a significant negative effect on labor distortion, rather than saving, and this distortion causes the crowding out of capital stocks. In our fertility/human capital endogenized model, there can inherently exist two equilibria (one with larger growth and low fertility, and the other with low growth and large fertility). If social security tends to induce the decrease in fertility of the poor class, it might be just because of the *accelerated* shift into a growth regime. [32] But then it might be rather a cause of less dispersed distribution. One recent stream of empirical research, Boldrin et al. (2005) or Ehrlich et al. (2007), also supports the argument that social security has a negative impact on fertility. This is one persuasive logic, because its positive

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[32] Of course we have to be careful about recent tendency where in developed countries the poor tend to bear less children than the rich. This is closely related with the selection of utility form.

disposable income effect, if exists, is more likely to push up the family into growth, as well as because social security seemingly gives the family less incentive for intergenerational private linkage, clearly a cause for less fertility. Therefore, the key question might be rather "Is social security more likely to suppress the investment in education or not, a trigger for regime change?" Before answering these issues, we want to be rigorous with respect to the definitions of social security, which is here divided majorly into three parts, inter-generational transfer in terms of certainty premium and actuarially fair insurance, and intra-generational redistribution. For example, Fuster (1999) considers progressive social security substantially as intra-generational redistribution from the rich class to the poor. Coate (1995) assumes an (intra-generational) asymmetric structure, consisting of public transfer, actuarially fair (or self-) insurance and private charity, between two classes, and extracts some inefficiency results as called the Samaritan's dilemma. Our article, focusing on intergenerational transfer, concludes that actuarially fair insurance, while enhancing parental old age utility, makes its long run effects on saving, fertility and education almost negligible in this two sided altruism, somewhat a different implication from the existing literature.

6. Discussions

Some crucial discussions regarding two-sided altruism are now stated.

Altruism, dynamic consistency and Pareto efficiency

Assume that parent and child hold following two sided altruistic utilities,

$$V_p(a_p,a_c)=v_p(a_p,a_c)+\lambda v_c(a_p,a_c) \text{ and } V_c(a_p,a_c)=v_c(a_p,a_c)+\lambda' v_p(a_p,a_c),$$

where $v_p(a_p,a_c)$ and $v_c(a_p,a_c)$ are individual indirect utilities of parent and child as functions of a_p and a_c, some actions implemented by parent and child respectively. These actions may be either simultaneous or sequential, but perfectly predictable anyway. If a_p and a_c are treated as independent actions, then ex-post Pareto optimality condition is $\frac{\partial v_p}{\partial a_p} / \frac{\partial v_p}{\partial a_c} = \frac{\partial v_c}{\partial a_p} / \frac{\partial v_c}{\partial a_c}$. Dynamic consistency $\lambda\lambda' = 1$ always assures this condition, regardless of how a decision flow between parents and children is ordered (parent first and child last, or vice versus). In other words, dynamic consistency basically eliminates negative externality (strategic distortion) which arises from sequential, asymmetric decision flows. Instead consider another case, as in Bernheim et al. (1985) or Bruce et al. (1990), that "children act first and parents second", in which bequest is treated as a retaliation strategy for controlling egoistic children. There child is assumed to be egoistic (i.e., $\lambda' = 0$) and take an action first (i.e., $a_p(a_c)$ as a function of a_c), and then Pareto optimality is attained only in a special case of $\frac{da_p}{da_c} = \frac{1}{\lambda} \frac{\partial v_p}{\partial a_c} / \frac{\partial v_c}{\partial a_p}$. In other words, the threat of retaliation for egoistic children does not necessarily attain Pareto efficiency. On the other hand, our model assumes an opposite decision flow, in which altruistic parents act first through fertility and education, and children second through compensation, but tied up with social norm λ'. Here, we regard the altruism toward parents λ' as a kind of self-enforcing restriction in a dynamic context, in which in case of breaking it children would have to put up with the same behavior from their own children. From game theoretic viewpoints, this kind of social norm is shown to be sustainable even for the case of negative compensation (i.e., positive bequest) in a steady state, while an "optimal" social norm could change over time especially in a transition process. However, dynamic consistency ($\lambda\lambda' = 1$) might not either achieve the highest growth or be the very strategy actually chosen by the generation. Furthermore, searching an appropriate λ' might not necessarily attain ex-post Pareto efficiency as of two periods later, because the number of restrictions λ and λ' (two) is less than the number of traded commodities h_t, n_t and ω_t (three), and because social norm restriction λ' itself does not actually contribute to the offset of strategic effects to grandchildren (offspring) regarding fertility and education decisions.

Bond-as-net-wealth controversy and Recardian equivalence

Effects of fiscal policy, known as bond-as-net-wealth controversy, had been discussed among Barro (1974, 1976), Feldstein (1976), Buchanan (1976) and Drazen (1978). Bruce et al. (1990) state, in a sequential decision flow where egoistic children act first and altruistic parents react, that Barro's debt neutrality proposition might not hold, because governmental redistribution from/to children cannot be fully manipulated by the children. In our two-sided altruistic model, the unfunded social security, financed as certainty premium transfer from the next generation (children), is clearly neutralized by perfectly substitutable private re-transfer from parents. However, as Drazen (1978) points out, "if the generation's horizon is shorter than that over which the government levies taxes to pay off the debt, the tax liabilities will not fully offset the present value of the interest payments" (i.e., a positive wealth effect arises), and in this generalized sense Ricardian equivalence does not hold. Now we will show that this debt neutrality does not necessarily hold in general, even for the case of a dynastic utility incorporating fully operative monetary transfer (bequest/compensation) mechanism.[33]

A simple dynastic utility is defined as:

[33] Barro's neutrality proposition basically holds for a dynastic utility incorporating one period for one life, and dynamic consistency.

$$\dot{V}_t \equiv u_t + \hat{\lambda}(n_t)u_{t+1} + \hat{\lambda}(n_t)\hat{\lambda}(n_{t+1})u_{t+2} + \cdots$$
$$= u_t + \lambda\pi_1 a(n_t)n_t u_{t+1} + \lambda^2 \pi_1{}^2 a(n_t)n_t a(n_{t+1})n_{t+1}u_{t+2} + \cdots$$

Compensation rate ω_t is determined at period $t-1$ as an implicit contract between the generation (vintage t) and its parental generation (vintage $t-1$). In order to internalize this intergenerational linkage with parental generation, define a "two-sided-altruistic" associated dynastic utility:

$$\hat{V}_t \equiv \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \lambda\pi_1 a(n_t)n_t u_{t+1} + \lambda^2\pi_1{}^2 a(n_t)n_t a(n_{t+1})n_{t+1}u_{t+2} + \cdots$$
$$= \hat{\lambda}'(n_{t-1})u_t^{(o)} + \dot{V}_t$$
$$= \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t + \lambda\pi_1 a(n_t)n_t \dot{V}_{t+1}$$
$$= \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t^{(y)} + (1-\lambda\lambda')\delta\pi_2 u_{t+1}^{(o)} + \lambda\pi_1 a(n_t)n_t \hat{V}_{t+1}$$

Assuming that the generation holds a bargaining power for determining compensation rate ω_t, and determines its life strategies in a recursive and backward induction, the value function, in which ω_t is internalized within altruistic dynastic utility $\hat{V}_t$, becomes:

$$V_t^*(H_t, n_{t-1}, S_{t-1}) = \max_{\{h_t, n_t, \omega_t, s_t\}} \begin{pmatrix} \hat{\lambda}'(n_{t-1})u_t^{(o)} + u_t^{(y)} + (1-\lambda\lambda')\delta\pi_2 u_{t+1}^{(o)} \\ + \lambda\pi_1 a(n_t)n_t V_{t+1}^*(H_{t+1}, n_t, \widetilde{S}_t) \end{pmatrix} \tag{6.1}$$

Here the objective function is $\hat{V}_t$. Consider the following (altruistic) dynastic utility maximizer:

$$\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\}_{t'=t}^{\infty} = \underset{\{h_{t'}, n_{t'}, \omega_{t'}, s_{t'}\}_{t'=t}^{\infty}}{\arg\max} \hat{V}_t \tag{6.2}$$

where $H'_{t+1} = A(H_t + \overline{H})h'_t$ and $S'_t \equiv D(H_t + \overline{H})(s'_t)^m$

$\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\}_{t'=t}^{\infty}$ is a set of subsequent life strategies, which maximizes the altruistic dynastic utility of generation t on the condition of current environment, H_t, n_{t-1}, S_{t-1}. Although, under dynamic consistency $(\lambda\lambda' = 1)$, $\{h'_{t'}, n'_{t'}, \omega'_{t'}, s'_{t'}\}_{t'=t}^{\infty}$ satisfies the dynastic equilibrium conditions in general, a necessary condition for dynamic efficiency,

$\{h'_t, n'_t, \omega'_t, s'_t\}$ does not necessarily coincide with the solution of (6.1). In case that the economy is *not* in steady state ($\{h_{t'}, n_{t'}, \omega_{t'}, s_{t'}\} \neq \{h, n, \omega, s\}$), (6.1) does not necessarily ensure, even under dynamic consistency, that equilibrium conditions of (6.2), especially those for H_{t+1} and N_{t+1}, are automatically satisfied, a cause of inefficiency (i.e., insufficient internalization), because they range over three periods. Therefore the generation's problem (6.1) is not equivalent with altruistic dynastic utility maximizer (6.2), as expected in a regular representative agent's model.

Assume that *one time* unfunded social security to the old aged $T_t (= \Omega_t^{ss})$ is financed by a sequence of taxation to subsequent young adult generations $\{\tau_{t'}\}_{t'=t}^{\infty}$. Then the above argument clarifies that how to organize $\{\tau_{t'}\}_{t'=t}^{\infty}$ *does* affect life strategies $\{h_t, n_t, \omega_t, s_t\}$, whether dynamic consistency holds or not. This is interesting, because intergenerational monetary transfer motivated on altruism is fully operative, and in addition children might be in a position to manipulate both private compensation to parents Ω_t^{fa} through altruism, and taxation on the future generations $\{\tau_{t'}\}_{t'=t+1}^{\infty}$ by exercising strategic effects to some (insufficient, though) extent. In this non-neutrality case, the net value of the bond (financing government expenditure T_t) might be either positive or negative, the latter of which could happen when $\{\tau_{t'}\}_{t'=t+1}^{\infty}$ is a trigger for regime change into a stagnant economy.[34] Here we define the net value of the bond issuance financing social security, as the change in the generation's representative utility ($\hat{V}_t$ here), given T_t and $\{\tau_{t'}\}_{t'=t}^{\infty}$. One exceptional case arises, for example, when T_t is also sequential $\{T_{t'}\}_{t'=t}^{\infty}$

[34] This is possible especially in dynastic utility approach. In two sided altruistic utility, the net value is more likely to be positive according to Drazen's argument.

and on a balanced path at $T_{t'} = \tau_{t'}$ ($t' = t, t+1, \cdots$). In this case, the growth rate of $T_{t'} (= \tau_{t'})$ does not matter in the determination of $\{h_t, n_t, \omega_t, s_t\}$. In other words, Recardian equivalence does hold, as far as taxation financing old age social security is involved in mandatory transfer from only neighbor (next), young adult generation (that is, $T_t = \tau_t$). Finally let's consider the case where intergenerational private bargaining between old parents and young adult is based on competition, not on altruism, as approximated in case A (saving only). In this case, basically Recardian equivalence does not hold, even if unfunded social security and taxation are well balanced at $T_{t'} = \tau_{t'}$ ($t' = t, t+1, \cdots$). The sign of bond's net value depends on initial state variables, H_t, n_{t-1}, S_{t-1}. Thus the question of "altruism or competition or the mix of them?" is crucial in this context, and should be left for future detailed empirical investigation.

Formulation of altruism, vertical vs. horizontal altruism

In this article, altruism is defined as a linearly-weighted and additively separable utility, which is an implicit restriction, exhibiting a decreasing return to scale of economy. This assumption is also the object for future econometric test. Theoretically, at least one-sided altruism (forward or backward) is essential and sufficient for the derivation of life strategies in a typical overlapping generation model, if dynamic consistency is implicitly assumed. Further positive implication of altruism can be found, for example, where there exists some production externality in a small community (firms, families, neighborhood or etc.) and this externality could be positively internalized, to some extent, within the altruistic utility framework. In this situation, especially in "horizontal" altruism, a member is obliged to take care of the marginal effects of other members' human/physical capital accumulation. Consequently the price of human capital becomes closer to a socially optimal level and for some condition the economy might be shifted from stagnant to growth regime with the increasing-return-to-scale property of knowledge.[35] Here one point, which should be paid careful attention for, is that the existence of consumption externality is not a sufficient condition for enabling this mechanism, but the way to control the consumption of other members (at intra-family, intra-firm or intra-community) through some investment/distributive channels needs to be precisely specified. The model of this article, with two-sided vertical altruism, satisfies it between two neighbor generations (young/old adulthood) within one identical family, while another important factor, specification of old-age capital investment & production, is missing. Therefore, the additional incorporation of old-age working activities is essential to examine the degree of intergenerational economic synergies caused by the internalization of intergenerational production externalities.

Empirical examples

We offer some empirical examples, which exhibit one support of two-sided altruistic model proposed in this article, using both a linear and a non-parametric regression model. Regression data between 1959 and 2000 were obtained from "IFS (International Financial Statistics) data", "Vital Statistics in Japan (Jinkou-doutai-chousa)", "SNA data", "Household Saving Statistics (Chochiku-doukou-chousa)" and "Population Statistics in Japan (Jinkou-suikei)". The following pair of regression equations are set.

$$
\begin{aligned}
y_i &= f(X_i) + \varepsilon_i \\
z_i &= g(X_i) + \eta_i \qquad (6.3) \\
X_i &= (x_{1i}, x_{2i}, \cdots, x_{Ki})
\end{aligned}
$$

Here, the sample size is T and $i(=1,2,\ldots,T)$ denotes each observation. The

[35] This is clear, for example, from the explanation by Obstfeld and Rogoff (1996) (section 7.3.1), in which the internalization of production externality under the AK model is described.

functions, $f(X_i)$ and $g(X_i)$ are non-parameterized $R^K \to R^1$ functions of dependent variables $X_i = (x_{1i}, x_{2i}, \cdots, x_{Ki})$, and y_i and z_i are explanatory variables. ε_i and η_i are assumed to follow a standard normal distribution, so $\varepsilon_i \sim N(0,\sigma_\varepsilon)$, $\eta_i \sim N(0,\sigma_\eta)$ and $Corr(\varepsilon_i, \eta_i) = \rho$. A following nonparametric estimator is defined for y_i, and so is similarly for z_i. Then fitted values ($\hat{y}_i = \hat{E}(y_i \mid X_i)$ and $\hat{z}_i = \hat{E}(z_i \mid X_i)$) and residuals ($\hat{\varepsilon}_t = y_i - \hat{y}_i$ and $\hat{\eta}_t = z_i - \hat{z}_i$) are obtained.

SVR: Saving rate (s_t)

FER: Total fertility rate (n_t)

OMR: Old aged (more than 64 years old) to middle age (15 to 64) population ratio ($1/n_{t-1}$)

SRR: Saving to revenue ratio (S_{t-1}/H_t)

DIR: Deposit interest rate (D_t)

GDP: Per capita GDP deflator (Price level discounted GDP per population, index for life "quality", H_t)

We set $y_i : SVR$ and $z_i : FER$, and assume the following four patterns as X_i:

Pattern 1: $X_i = (OMR, SRR, DIR, GDP)$

Pattern 2: $X_i = (SRR, DIR, GDP)$

Pattern 3: $X_i = (OMR, DIR, GDP)$

Pattern 4: $X_i = (DIR, GDP)$

Two regression models are constructed as follows.

Linear regression model

$$y_i = \beta_0 (+\beta_1 OMR_i + \beta_2 SRR_i) + \beta_3 DIR_i + \beta_4 GDP_i + \varepsilon_i$$

$$z_i = \gamma_0 (+\gamma_1 OMR_i + \gamma_2 SRR_i) + \gamma_3 DIR_i + \gamma_4 GDP_i + \eta_i$$

Non-parametric (one-leave-out) regression model

$$y_i = f((OMR_i, SRR_i,)DIR_i, GDP_i) + \varepsilon_i$$

$$z_i = g((OMR_i, SRR_i,)DIR_i, GDP_i) + \eta_i$$

For the null hypothesis $H_0 : Cov(\varepsilon_i, \eta_i) = 0$ (stochastic independence between ε_i and η_i), the test statistics for testing asymmetric information is based on Chiappori and Salanié (2000).

$$W = \left(\sum_{i=1}^{T} (\hat{\varepsilon}_i - \bar{\hat{\varepsilon}})(\hat{\eta}_i - \bar{\hat{\eta}})\right)^2 / \sum_{i=1}^{T} (\hat{\varepsilon}_i - \bar{\hat{\varepsilon}})^2 (\hat{\eta}_i - \bar{\hat{\eta}})^2$$

$\bar{\hat{\varepsilon}}$ and $\bar{\hat{\eta}}$ are sample means of $\hat{\varepsilon}_i$ and $\hat{\eta}_i$, respectively. A benchmark non-parametric bandwidth policy for x_k was taken at $h_k^0 = c(4/3)^{1/5} T^{-1/5} \hat{\sigma}_{x_k}$ ($\hat{\sigma}_{x_k}$ is a sample variance of x_k.), where $c = 10.0$, 1.0, and 0.1. Regression results are summarized in Table 4-a and b. In a linear regression case (Table 4-a, pattern 1), SRR (saving to revenue ratio), rather than OMR (the old age population to the middle ratio), is 1% significant for y : SVR (saving rate), while OMR is about 20% significant. For z : FER (fertility rate), SRR is not significant, while OMR is about 1% significant. On the other hand, non-parametric regression results (Table 4-b) show that conditional correlation between y and z exists in pattern 3 and 4 for both $c = 1.0$ and 0.1, therefore that SRR, rather than OMR, might be a common causality between y and z. The weakness of OMR as a common causality could be a consequence of nearly adjusted-fertility neutralization, $\varepsilon \cong \sigma$. To summarize, both relative population (OMR) and saving to revenue ratio (SRR) are significant within the constructed model, especially SRR could be a common causality between fertility and saving rate. These results partially support this two-sided altruistic setting, which internalizes the external circumstances from parental generation, relative population and saving to revenue ratio, into state variables. The additional inclusion of another explanatory variable, per-capita social security expenditure (SSE, say) exhibits that both y and z have negative slopes on SSE with significance. This supports the arguments of Boldrin et al. (2005) and Ehrlich et al. (2007). Since SSE includes all kinds of social security (social insurance, public aid, social welfare, public health), we need to investigate each effect, parts by parts, with more careful examination of causality (which is a cause or a result?), and this task is left for the future.

Policy implications

Wigniolle (2002) emphasizes, also based on two sided model, the effectiveness of three policies, fertility control, education subsidies and old-age pension system, by means of taking arbitrage among financing them. Let $\tilde{\omega}$, $\tilde{h}$, $\tilde{n}$ be proportional taxes (subsidies for the negative) on compensation, education and children (ω_t, h_t and n_t), respectively. Then ω_t, h_t and n_t in (2.3-d) are replaced with $\omega_t \to \omega_t(1+\tilde{\omega})$, $h_t \to h_t(1+\tilde{h})$ and $n_t \to n_t(1+\tilde{n})$, and thereby the effective weight of altruism are transformed to $\lambda' \to \lambda'/(1+\tilde{\omega})$, $\lambda \to \lambda/(1+\tilde{h})$ and $\lambda \to \lambda/(1+\tilde{n})$, respectively. The adjustment of λ' proves to play a crucial role on efficiency, rather than on growth. In the light of the discussions regarding bond-as-net-wealth controversy, these taxes are hopefully to be financed within the underlying period t (or at most over two periods), because of insufficient internalization problem. Then:

$$v\tilde{n}n_t + \{(1+\tilde{n})(1+\tilde{h})-1\}h_t n_t + \tilde{\omega}\omega_t = 0 \tag{6.4}$$

Assuming, as in section 4, that altruism-adjusted aggregate utility ($\ddot{u}_{t+1} \equiv (n_t)^{1-\varepsilon} u_{t+1}$) represents generation $t+1$, some further policy implications are summarized as follows.

(a) Old parents and young adult children are motivated by altruism. (e.g., case B and C)

Unfunded social security for the old aged, $\overline{\Omega}_t^{ss} = \theta H_t$, as an income tax on the middle age, does not make any sense, but a compensation tax $\tilde{\omega}\omega_t$ is effective as an adjustment measure of λ'. Two policies aimed at pushing into a growth and at attaining Pareto efficiency might not necessarily be consistent with each other. For example, under a growth regime and $\varepsilon > \sigma$, Pareto improving taxation (subsidy) policy could be found, where it could be more altruistic ($\lambda\lambda' > 1$), but yet might be growth decelerating. Actuarially fair aspect of old age insurance is recommended, because it is supposed not to affect a lot education, fertility and saving (h_t, n_t and s_t) at least in an almost stable state, while surely enhancing the old age utility.

(b) Old parents and young adult children are competitive. (e.g., case A)

Contrary to (a), financing the old age social security by an income tax makes sense, good or evil, while a compensation tax is not effective. Under a growth regime, the subsidy for fertility $\tilde{n}(<0)$ might not be, unfortunately, Pareto efficiency improving. For $\sigma < 1$ ($\sigma > 1$), actuarially fair insurance increases (decreases) saving, while suppresses (stimulates) education and fertility through a negative (positive) income effect.

7. Final remarks- summary

Thus the main idea this article is to discuss the interrelation among the attribute of intergenerational linkage, social norm, intergenerational equity, growth, and the effect of social security, with a model of two sided altruism. The model incorporates three state variables (own human capital, parental saving, and relative population to old parents) and four control variables (education, fertility, saving, and compensation). Policy functions of each generation are endogenously created by means of iterative backward conjecture. There does not exist such literature, that it comprehensively treats these issues in a simplest way and extracts some crucial implications.

This article based on theoretical frameworks by Ehrlich and Lui (1991) (human capital-fertility endogenized overlapping generations model), Becker, Murphy and Tamura (1990) (elasticity of altruism per child), and Abel (1987), Kimball (1987), Altig and Davis (1993) and many others (two sided altruism). It also takes careful account of the timing of decision making and the strategic effects on offspring generations, as well as the external effect from parental generation's circumstances. Specifically the model is described as equations (2.1) and (2.2). The model setting enables some extensive arguments related with so called the Samaritan's dilemma (e.g., Bernheim, Shleifer and Summers (1985), Bruce and Waldman (1990), Coate (1995)), bond-as-net-wealth controversy (e.g., Barro (1974), Feldstein (1976)), and social security issues (e.g., Kotlikoff et al. (1981, 1986), Altig and Davis (1991), Fuster (1999), Boldrin, De Nardi and Jones (2005)).

In section 3, 4, 5, we clarified the dynamics of this two-sided altruistic model, and analyzed how the weights of altruism, as a sort of social norm, affect growth, efficiency and the effect of actuarially fair insurance. Especially the ex-post Pareto efficiency criterion as of two periods later is a novel concept in the literature, and it enables the relation between efficiency and growth in a dynamic context to be intuitively captured. In addition, defining a normalized value function is shown to explain how the social norm could be "optimally" generated in the long run. It is clarified how the backward altruism affects the generation's strategies from game-theoretic viewpoints. Some implications regarding the Samaritan's dilemma and bond-as-net-wealth controversy are also new.

Basically we are not intending to insist the stated implications as "absolutely true" or "false" ones. We have just offered one extensive, but elegant tool for linking growth and efficiency under social norm restrictions. The analyses are successful, because the model is adjusted so that three essential factors of intergenerational trade (education, fertility and compensation) are, in appropriate timings, incorporated within the OLG (three stage life cycles) economy under two restrictions (altruism for parents and children). Some existing papers may have the same concerns, but still limited and insufficient, because they consider only one period for one life or do not assume two-sided altruism or do not care about fertility. Specifically, the propositions and the economic logics behind them are summarized as following.

Growth and ex-post Pareto efficiency criterion as of two periods later

We showed some efficiency relations among dynamic consistency, initial regimes (stagnant or growth), elasticity of altruism, and relative risk aversion. As for the calibration results with a particular set of parameters are concerned, egoistic attitude ($\lambda\lambda'<1$) accelerates growth, while a potential possibility for ex-post Pareto efficiency is left for the case of altruistic attitude ($\lambda\lambda'>1$) and $\varepsilon > \sigma$.

Creation of social norm

It was analyzed how the altruism toward parents, λ' , does influence the generation's life strategies, and it is shown that, in spite of its direct negative income effect, the optimal point is likely to be positive, because the policy (reaction) functions from next generations are *explicit* functions of social norms.

Neutrality proposition of actuarially fair insurance on fertility, education and saving

As for social security, one proposition regarding the neutrality of actuarially fair insurance on fertility, education and saving is derived from steady states restrictions (in stead of regular reaction functions). Under this assumption, the existence of actuarially fair insurance might not affect so drastically fertility and education (and saving), while only compensation is sensitive to the difference from self insurance. This outcome arises from it that the incentive (or disincentive) for investing these three factors might well be offset through adjustment in compensation, regardless of the values of the combination of altruism (λ, λ') and other system parameters.

The Samaritan's dilemma

In general, dynamic consistency ($\lambda\lambda'=1$) eliminates negative externality (strategic distortion), which comes out from a sequential asymmetry of decision flows. However, in our model with infinite time horizon, general inefficiency arises even under dynamic consistency, because the number of social norm restrictions is less than that of intergenerational traded goods, and because backward altruism toward parents does not actually contribute to the counterbalance of the strategic effects through education and fertility onto grandchildren within the corresponding representative utility, a failure of envelope theorem.

Debt neutrality proposition (Recardian equivalence)

The model is related to so called bond-as-net-wealth controversy. We showed that this debt neutrality does not necessarily hold even for dynastic utility, especially in an unsteady (or a transition) process, because the underlying generation fails in complete internalization, even under dynamic consistency and fully operative transfer. This point is noteworthy, because the generation can be manipulative both toward parents fully through monetary transfer based on altruism, and toward children insufficiently through education and fertility.

The numerical simulation was implemented rigorously according to equations (2.1) and (2.2), where reaction (policy) functions endogenously spring out from backward induction. We verified that general properties examined by Becker et al. (1990) or Ehrlich et al. (1991) basically hold for this two sided altruism, as well as some others like irreversible hysteresis aspect or dynamic fluctuation. On the other hand, because of technical problems, some computation results were derived instead simply from steady states restrictions, which might not necessarily be the outcomes the economy eventually arrives at. (e.g., "Examples" in section 3 and 5) This point is left for future analysis. However, these steady states restrictions are shown to derive a crucial implication regarding the neutrality of actuarially fair insurance on saving, education and fertility.

We also tested the model with the aggregate data of Japanese house hold economy, and showed, for example, that saving to revenue ratio is a common cause for saving rate and fertility, which partially supports my two-sided altruistic setting. It is well known that there are some big trends in developed countries, for example decreasing fertility and saving rate, increasing social security expenditure, or etc., so a sort of co-linearity problem arises. Our regression results show that these causalities could be detected even after removing these co-linear trends by way of non-parametric regression methods.

Our current hypothesis is that the intergenerational bargaining between young adult children and old parents would consist of *the mixture of competition and altruism*. In order to verify the hypothesis, we would need, preferably, a large scale household panel survey data, which contain intra-family private compensation and receipts/payments of social security, and upon which it is possible to estimate parametrically the combination of altruism toward children and parents (λ and λ'), as well as other parameters.

Finally, the points we have learned from the above discussions are summarized as following.

a. The optimal choice of backward altruism toward parents (λ') for attaining the best growth, efficiency and normalized representative utility, respectively, might not be consistent with each other. As a consequence, the resultant optimal policies might be different. Thus the conventional theories and corresponding implications might be more suitably reconsidered and reexamined within the proposed two sided altruistic model.

b. In discussing the effects of social security, we had better be a little more rigorous with the definitions of social security. As far as the role of actuarially fair insurance is concerned, it does not affect a lot saving, education and fertility.

c. We would need to specify the attribute of intergenerational linkage between young adults and old parents, e.g., competition, altruism or a mix of them. In this sense, further detailed empirical investigation regarding it and regarding the estimation of utility functions and system parameters, would be needed for the construction of appropriate social security scheme.

d. The role of two sided altruism is limited and less appealing, as far as the implications as an internalization device of vertical/horizontal production externalities are not fully discussed.

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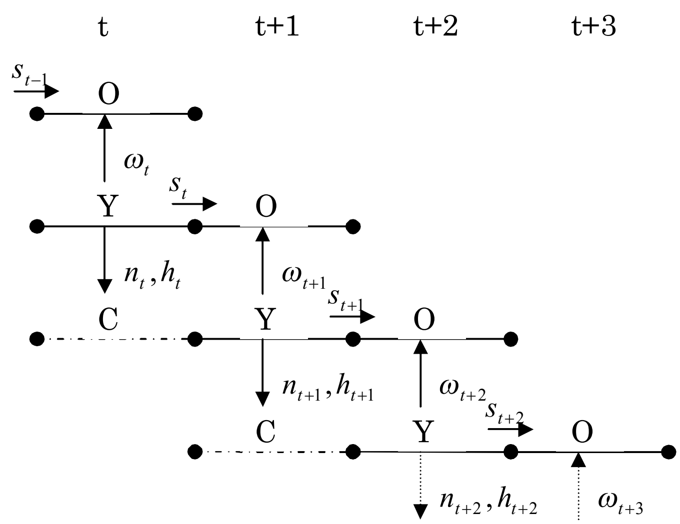
"Population Statistics in Japan (Jinkou-suikei)", Statistics Bureau, Director-General for Policy Planning & Statistical Research and Training Institute, Ministry of Internal Affairs and Communications

"IFS (International Financial Statistics) data", International Monetary Fund (IMF), August 2006.



Case C

Figure 1 Specification of Intergenerational Linkage (Case C)

σ	ε	δ	v	$\overline{H}$	D	m	δ'	π_1	π_2
0.5	0.5	0.5	0.1	1.0	2.0	0.5	$\delta^{0.5}$	1.0	1.0

h_t	$h_{\min} = 0 \leq h_t \leq 1.0 = h_{\max}$
s_t	$s_{\min} = 0 \leq s_t \leq 0.5 = s_{\max}$
n_t	$n_{\min} = 0.5 \leq n_t \leq 2 = n_{\max}$ (0.5, 1.0, 1.5, 2.0)
ω_t	$\omega_{\min} = -0.5 \leq \omega_t \leq 0.5 = \omega_{\max}$ (only in case B and C)

Table 1
Default Values of Parameter Settings

Table 2 Equilibria under steady states restriction (Case B)

A=5, sigma=0.5, pai2=1.0 ＊Normalized value = Value/(1+lamda+lamdaprime)

<u>No uncertainty case</u>

I. lamda=0.5, lamdaprime=2.0

H1=0

n	h	ω	Value	Normalized*
0.5	0.37	0.38	3.9504	1.1287
1.0	0.10	0.40	4.0790	1.1654
1.5	0.01	0.42	4.2064	1.2018
2.0	0.00	0.40	4.3187	1.2339

H1=10

n	h	ω	Value	Normalized*
0.5	0.49	0.35	12.3155	3.5187
1.0	0.22	0.34	11.9565	3.4162
1.5	0.13	0.33	11.5939	3.3125
2.0	0.09	0.31	11.2278	3.2080

II. lamda=0.5, lamdaprime=0.5

H1=0

n	h	ω	Value	Normalized*
0.5	0.47	0.16	3.1718	1.5859
1.0	0.15	0.17	3.2822	1.6411
1.5	0.05	0.18	3.3920	1.6960
2.0	0.00	0.19	3.5010	1.7505

H1=10

n	h	ω	Value	Normalized*
0.5	0.58	0.15	9.8466	4.9233
1.0	0.26	0.14	9.5403	4.7702
1.5	0.16	0.13	9.2323	4.6161
2.0	0.11	0.12	8.9209	4.4604

III. lamda=2.0, lamdaprime=0.5

H1=0

n	h	ω	Value	Normalized*
0.5	0.70	0.04	6.3867	1.8248
1.0	0.27	0.04	6.6528	1.9008
1.5	0.13	0.04	6.9181	1.9766
2.0	0.06	0.04	7.1823	2.0521

H1=10

n	h	ω	Value	Normalized*
0.5	0.80	0.03	19.5742	5.5926
1.0	0.37	0.03	18.8470	5.3849
1.5	0.23	0.03	18.1170	5.1763
2.0	0.16	0.03	17.3832	4.9666

IV. lamda=2.0, lamdaprime=2.0

H1=0

n	h	ω	Value	Normalized*
0.5	0.66	0.07	6.5561	1.3112
1.0	0.25	0.07	6.8202	1.3640
1.5	0.12	0.07	7.0833	1.4167
2.0	0.05	0.07	7.3457	1.4691

H1=10

n	h	ω	Value	Normalized*
0.5	0.76	0.06	20.1430	4.0286
1.0	0.35	0.06	19.4192	3.8838
1.5	0.22	0.06	18.6926	3.7385
2.0	0.15	0.06	17.9639	3.5928

Table 3 Effects of two insurances on life strategies under seady states restriction

Case C (Both saving and compensation economy)
A=5.0, pai2=0.5, lamda=1.0, lamdaprime=1.0 S0=0.5(H1+Hbar)

(1) Actuarially fair insurance case

sigma=0.5

H1=0

n	h	ω	s	Value
0.5	0.28	−0.65	0.25	−1.8390
1.0	0.14	−0.27	0.13	−2.6707
1.5	0.06	−0.15	0.09	−3.2406
2.0	0.01	−0.10	0.08	−3.6846

H1=10

n	h	ω	s	Value
0.5	0.42	−0.71	0.25	14.2315
1.0	0.26	−0.35	0.14	11.3461
1.5	0.17	−0.23	0.10	9.4732
2.0	0.12	−0.17	0.08	7.9584

sigma=1.0

H1=0

n	h	ω	s	Value
0.5	0.25	−0.35	0.20	−2.6920
1.0	0.12	−0.14	0.12	−3.3993
1.5	0.05	−0.08	0.09	−4.0292
2.0	0.01	−0.04	0.07	−4.5846

H1=10

n	h	ω	s	Value
0.5	0.39	−0.41	0.21	7.9396
1.0	0.25	−0.22	0.13	7.2308
1.5	0.17	−0.16	0.09	6.5651
2.0	0.13	−0.14	0.08	5.8504

sigma=2.0

H1=0

n	h	ω	s	Value
0.5	0.17	−0.14	0.15	−4.5959
1.0	0.09	−0.06	0.11	−5.3826
1.5	0.04	−0.02	0.08	−6.3600
2.0	0.01	−0.01	0.07	−7.3745

H1=10

n	h	ω	s	Value
0.5	0.33	−0.20	0.16	3.6691
1.0	0.23	−0.13	0.11	3.6834
1.5	0.18	−0.11	0.09	3.6462
2.0	0.14	−0.10	0.07	3.5129

(2) Self insurance case

sigma=0.5

H1=0

n	h	ω	s	Value
0.5	0.27	−0.70	0.26	−1.9195
1.0	0.14	−0.29	0.13	−2.7459
1.5	0.06	−0.17	0.09	−3.3146
2.0	0.01	−0.12	0.08	−3.7592

H1=10

n	h	ω	s	Value
0.5	0.42	−0.75	0.26	13.9766
1.0	0.26	−0.37	0.14	11.1186
1.5	0.18	−0.25	0.10	9.2544
2.0	0.12	−0.19	0.08	7.7421

sigma=1.0

H1=0

n	h	ω	s	Value
0.5	0.25	−0.35	0.20	−2.9878
1.0	0.12	−0.14	0.12	−3.6444
1.5	0.05	−0.08	0.09	−4.2518
2.0	0.01	−0.04	0.07	−4.7937

H1=10

n	h	ω	s	Value
0.5	0.39	−0.41	0.21	7.6438
1.0	0.25	−0.22	0.13	6.9858
1.5	0.17	−0.16	0.09	6.3425
2.0	0.13	−0.14	0.08	5.6412

sigma=2.0

H1=0

n	h	ω	s	Value
0.5	0.17	−0.10	0.15	−5.5792
1.0	0.09	−0.02	0.10	−6.1855
1.5	0.04	0.00	0.08	−7.0923
2.0	0.01	0.01	0.07	−8.0684

H1=10

n	h	ω	s	Value
0.5	0.33	−0.15	0.15	3.5744
1.0	0.23	−0.10	0.11	3.5975
1.5	0.18	−0.10	0.09	3.5580
2.0	0.14	−0.09	0.07	3.4184

Table 4-a Linear regression model

			Const	OMR	SRR	DIR	GDP	R2	W
Pattern 1	y (SVR)		0.1454	0.3733	−0.1001	0.0056	0.1073	0.6213	7.0663
		t-statistics	4.4182	1.2668	−4.0696	1.2436	3.2466		
	z (FER)		2.4969	−4.2271	0.0169	0.0002	−0.2275	0.8276	
		t-statistics	12.9803	−2.4543	0.1179	0.007	−1.1774		
Pattern 2	y (SVR)		0.1676	−	−0.0795	0.0028	0.1258	0.6049	3.6330
		t-statistics	5.9696	(−)	−4.2747	0.7038	4.2029		
	z (FER)		2.2454	−	−0.2163	0.0322	−0.4362	0.7995	
		t-statistics	12.9615	(−)	−1.8858	1.3221	−2.3625		
Pattern 3	y (SVR)		0.1248	−0.4198	−	0.0084	0.073	0.4518	4.8626
		t-statistics	3.2328	−1.5997	(−)	1.5841	1.9243		
	z (FER)		2.5003	−4.0928	−	−0.0003	−0.2217	0.8275	
		t-statistics	13.329	−3.2099	(−)	−0.0111	−1.2022		
Pattern 4	y (SVR)		0.072	−	−	0.0152	0.0221	0.4148	7.2234
		t-statistics	3.5321	(−)	(−)	4.716	1.0492		
	z (FER)		1.9851	−	−	0.066	−0.7184	0.7808	
		t-statistics	18.3755	(−)	(−)	3.8653	−6.4352		

R2: Coefficient of determination W: Test statistics for conditional correlation

Table 4-b Non-parametric regression model

		c=10.0	c=1.0	c=0.1
Pattern 1	W	12.59	1.1573	2.073
Pattern 2	W	12.9827	1.2252	1.6728
Pattern 3	W	13.1611	3.9267	3.6851
Pattern 4	W	13.5346	6.8422	4.0391

W: Test statistics for conditional correlation between y (FER) and z (SVR)